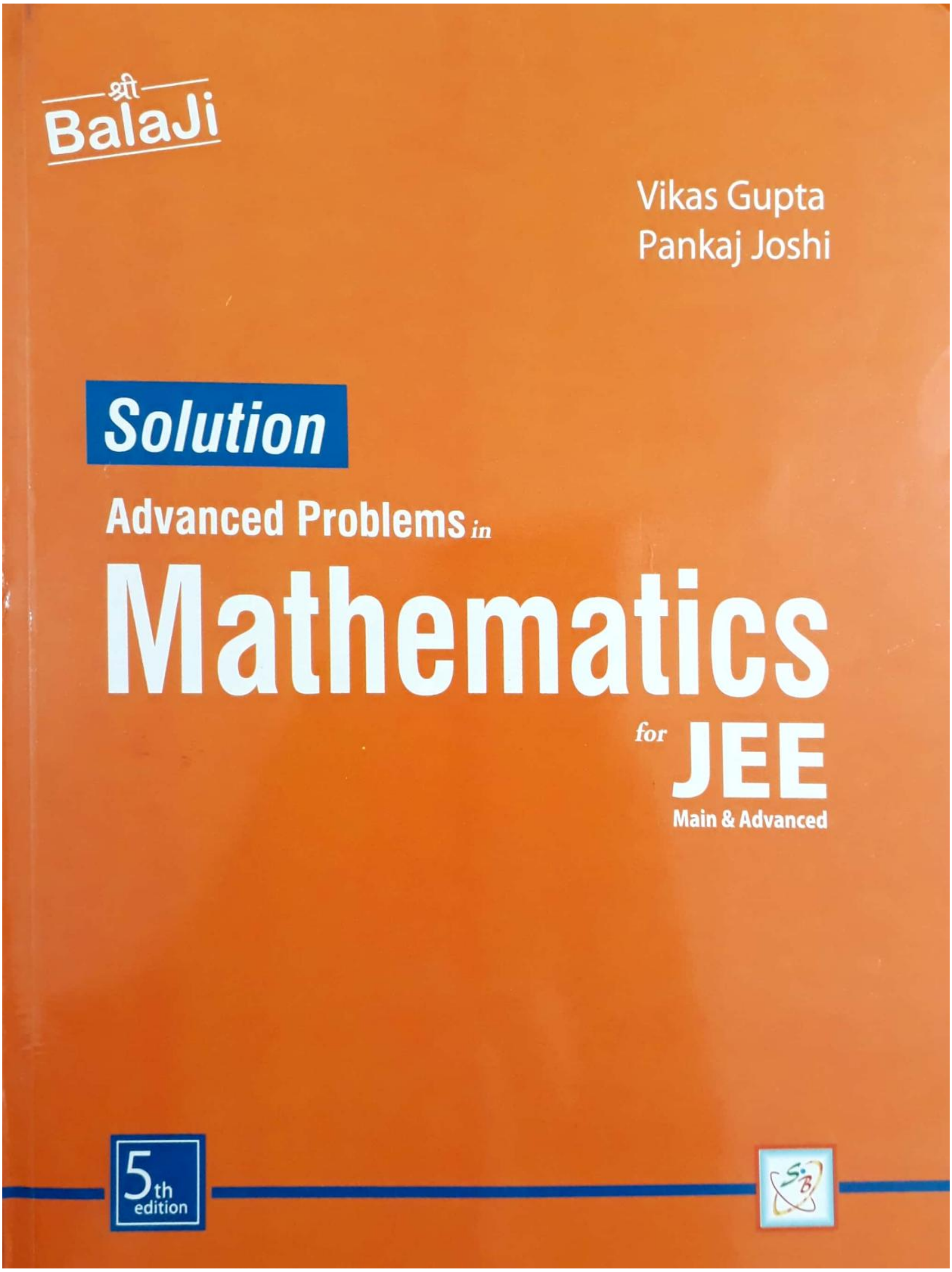
Balaji

Solution to Advanced Problems in Mathematics for IIT JEE

Main and Advanced

by

Vikas Gupta and Pankaj Joshi


श्री BalaJi
Vikas Gupta
Pankaj Joshi
Solution
Advanced Problems in
Mathematics
for JEE
Main & Advanced
5th edition

श्री BalaJi

SOLUTION *to* Advanced Problems in MATHEMATICS *for* JEE (MAIN & ADVANCED)

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Chapter 1 – Function

1 FUNCTION

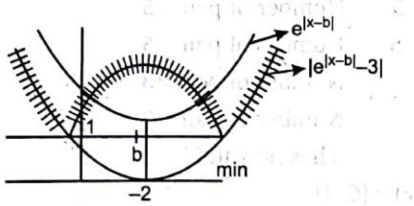
Exercise-1 : Single Choice Problems

1. $f(x) = \log_2(2 - 2\log_{\sqrt{2}}(16\sin^2 x + 1))$

$0 \le \log_{\sqrt{2}}(16\sin^2 x + 1) \le \log_2 17 \quad \Rightarrow 2 - 2\log_2 17 \le 2 - 2\log_{\sqrt{2}}(16\sin^2 x + 1) \le 2$

$\Rightarrow \quad 0 < 2 - 2\log_{\sqrt{2}}(16\sin^2 x + 1) \le 2 \quad \Rightarrow f(x) \le 1$

2. For any $b \in R$ $e^{|x-b|}$ is



$|e^{|x-b|} - a|$ has four distinct solutions $a > 3$ so $a \in (3, \infty)$

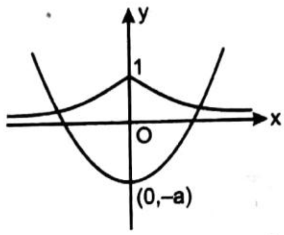
3. Domain $= [-1, 1]$ and both are increasing functions.

$\therefore \quad x = -1$, we get minimum value & $x = 1$, we get maximum value.

$$\left[-\frac{\pi}{4} - \frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{4}\right] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

4. $\left(2^{2x^2+2y} - 2^{2x+2y^2}\right)^2 = 1 - 2^{2x^2+2y^2+2x+2y+1} \ge 0$

$$\Rightarrow \qquad \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \le 0$$

5. 

7. $\sec^{-1}\left(-\frac{5}{2}+\frac{2}{2(x^2+2)}\right)$ $\quad x^2+2\geq 2$

$=\sec^{-1}\left(-\frac{5}{2}+\frac{1}{(x^2+2)}\right)$ $\quad \left(\frac{1}{x^2+2}\leq\frac{1}{2}\right)$

$\leq\sec^{-1}(-2)=\pi-\sec^{-1}(2)$ $\quad \left(-\frac{5}{2}+\frac{1}{x^2+2}\leq-\frac{5}{2}+\frac{1}{2}=-2\right)$

$=\frac{2\pi}{3}$

8. $f'(x)=x^2+ax+b$ is injective if $D\leq 0$

$a^2-4b\leq 0$

If	$a=1, b=1,2,3,4,5$	Number of pair $=5$
	$a=2, b=1,2,3,4,5$	Number of pair $=5$
	$a=3, b=3,4,5$	Number of pair $=3$
	$a=4, b=4,5$	Number of pair $=2$
	$a=5$	b has no value

9. $f(x)=\log_x[x]$ $\Rightarrow f(x)\in[0,1]$

$g(x)=|\sin x|+|\cos x|$

$\Rightarrow$ $g(x)\in[1,\sqrt{2}]$

10. $f(x)=2x^3-3x^2+6$

$f'(x)=6x^2-6x\geq 0.$ $\quad\Rightarrow x\in[1,\infty)$

and $f(x)\in[5,\infty)$

11. $0\leq\{x\}<1$

$\{x\}(\{x\}-1)(\{x\}+2)\geq 0$

$\Rightarrow$ $\{x\}=0$ $\Rightarrow x\in z$

14. $1+\sin^2 x\in[1,2]$

$\frac{1}{1+\sin^2 x}\in\left[\frac{1}{2},1\right]$

$\sin^{-1}\left(\frac{1}{1+\sin^2 x}\right)\in\left[\frac{\pi}{6},\frac{\pi}{2}\right]$

$\frac{K\pi}{6} \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \qquad K \in [1, 3]$

15. $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$

Put $x = y = 0$

$f(0) = [f(0)]^2 - f(a)f(a)$

$\Rightarrow \quad f(a) = 0 \qquad [\because f(0) = 1]$

Put $x = a$ and $y = x$

$f(a-x) = f(a)f(x) - f(0)f(a+x)$

$\Rightarrow \quad -f(a-x) = f(a+x) \;\Rightarrow f(2a-x) = -f(x)$

18. $f(x) = 4x - x^2 = y$

$x^2 - 4x + y = 0$

$f^{-1}(x) = 2 - \sqrt{4-x}$

19. $[5\sin x] + [\cos x] = -6$

$\Rightarrow \quad -1 \le \cos x < 0 \quad$ and $\quad -5 \le 5\sin x < -4$

$-1 \le \sin x < -\frac{4}{5}$

20. $f(x) = ax + \cos x$

$f'(x) = a - \sin x$

if $f(x)$ is invertible, then

$f'(x) \ge 0$ or $f'(x) \le 0$

$\Rightarrow \quad a \ge 1$ or $a \le -1$

21. $f(x) = [1 + \sin x] + \left[2 + \sin\frac{x}{2}\right] + \left[3 + \sin\frac{x}{3}\right] + \ldots + \left[n + \sin\frac{x}{n}\right]$

$= (1 + 2 + 3 + \ldots + n) + [\sin x] + \left[\sin\frac{x}{2}\right] + \left[\sin\frac{x}{3}\right] + \ldots + \left[\sin\frac{x}{n}\right]$

22. $y = \frac{x^2 + ax + 1}{x^2 + x + 1}$

$(y-1)x^2 + (y-a)x + (y-1) = 0$

$D \ge 0$

$(y-a)^2 - 4(y-1)^2 \ge 0$

$-3y^2 + y(8-2a) + a^2 - 4 \ge 0 \;\forall\; y \in R$

Not possible

23. $f(x) = [x] + [-x]$

$f(x)=\begin{cases}0 & x\in I\\ -1 & x\notin I\end{cases}$

$g(x)=\{x\}$

$h(x)=f[g(x)]=f(\{x\})$ $\qquad$ $\{x\}=0 \quad x\in I$

$\{x\}=\{x\} \quad x\notin I$

$h(x)=\begin{cases}f(0) & x\in I\\ f(\{x\}) & x\notin I\end{cases} \Rightarrow h(x)=\begin{cases}0 & x\in I\\ -1 & x\notin I\end{cases}$

Hence, the option (b).

24. $f(x)=\left[\frac{x}{15}\right]\left[-\frac{15}{x}\right] \qquad x\in(0,90)$

$0\le x<15$	$f(x)=0$
$15\le x<30$	$f(x)=-1$
$30\le x<45$	$f(x)=-2$
$45\le x<60$	$f(x)=-3$
$60\le x<75$	$f(x)=-4$
$75\le x<90$	$f(x)=-5$

Total integers in range $f(x)=\{0,-1,-2,-3,-4,-5\}$

25. $g(x)=\dfrac{1}{f(|x|)}$

$g(x)\Rightarrow$ even functions $\Rightarrow$ symmetric about y-axis

$\Rightarrow \quad x\to\infty \quad f(x)\to 0$

at $x=x_1 \qquad f(x)=0 \Rightarrow g(x_1)\to\infty$

26. Homogeneous function $\Rightarrow f(tx,ty)=t^n f(x,y)$

27. $f(x)=\begin{bmatrix}2x+3 & x\le 1\\ a^2x+1 & x>1\end{bmatrix}$

For $x\le 1 \qquad f(x)\le 5$

So for range of $f(x)$ to be R.

$\Rightarrow \quad a^2+1\le 5$ and $a\ne 0$

$\Rightarrow \quad a\in[-2,2]$

Hence, $a=\{-2,-1,1,2\}$

28. $\log_{1/3}(\log_4(x-5))>0$

$0<\log_4(x-5)<1$

$1<x-5<4$

$6<x<9$

29. $f(x)=\log_2\left(\dfrac{4}{\sqrt{2+x}+\sqrt{2-x}}\right); -2\le x\le 2$

$\sqrt{2+x}+\sqrt{2-x}=y$

$4+2\sqrt{4-x^2}=y^2$

$y\in[2,2\sqrt{2}]$

Range $f(x)=\left[\log_2\frac{4}{2\sqrt{2}},\log_2\frac{4}{2}\right]$

$f(x)$ lies between $\left[\frac{1}{2},1\right]$

30. $|x^2+5x|+|x-x^2|=|6x| \Rightarrow |x^2+5x|+|x-x^2|=|(x^2+5x)+(x-x^2)|$

$|a|+|b|=|a+b| \Rightarrow ab\geq 0$

$(x^2+5x)(x-x^2)\geq 0$

$x(x+5)\cdot x(x-1)\leq 0 \Rightarrow -5\leq x\leq 1$

31. $f(x)+f\left(\frac{1}{x}\right)=f(x)f\left(\frac{1}{x}\right)$

$\Rightarrow\quad f(x)=1\pm x^n$

$f(2)=33 \Rightarrow n=5$

Hence, $f(x)=1+x^5$

Here, $f(x)+f(-x)\neq 0.$

Hence not an odd function.

32. $g(x)=\frac{\sin x+\sin 7x}{\cos x+\cos 7x}+|\sin x|=\frac{2\sin 4x\cos 3x}{2\cos 4x\cos 3x}+|\sin x|$

$=\tan 4x+|\sin x|$

$g(x)$ period $=\pi$

33. $f(x)=\begin{cases}\frac{x-1}{2} & x=\text{odd}\\ -\frac{x}{2} & x=\text{even}\end{cases}$ $\qquad f(x):N\to Z$

Let $x=\text{odd}=(2n+1);n>0$

$f(x)=\frac{2n+1-1}{2}=n \Rightarrow$ +ve integer

Let $x=\text{even}=2m;m>0$

$f(x)=-\frac{2m}{2}=-m \quad\Rightarrow$ –ve integer

$\Rightarrow$ Range = codomains $\Rightarrow$ onto and clearly $f(x)$ is one-one function.

Hence, bijective.

34. $y=\frac{2^{x+1}-2^{1-x}}{2^x+2^{-x}}=\frac{2^{2x+1}-2}{2^{2x}+1}$

$y(2^{2x}+1)=(2^{2x}-1)2$

$2^{2x}\cdot y+y=2^{2x}\cdot 2-2$

$2^{2x}(y-2)=-(2+y)$

$2^{2x}=\dfrac{(y+2)}{(2-y)}$

$2x=\log_2\dfrac{(2+y)}{(2-y)}$

$x=\dfrac{1}{2}\log_2\dfrac{(2+y)}{(2-y)}$

$f^{-1}(x)=\dfrac{1}{2}\log_2\dfrac{(2+x)}{(2-x)}$

35. $\sqrt{|y|}=x$ $(\because x>0)$

$\Rightarrow\ |y|=x^2$

$\Rightarrow\ y=x^2 \quad y\ge 0$

$y=-x^2 \quad y<0$

36. $f(x)=\log_{[x]}(9-x^2)$

Domains = $\left.\begin{matrix}9-x^2>0\\ [x]>0 \text{ and } [x]\neq 1\end{matrix}\right\} \Rightarrow x\in[2,3)\Rightarrow f(x)=\log_2(9-x^2)$

Range $=(-\infty,\log_2 5]$

37. Gives $e^x+e^{f(x)}=e$

$e^{f(x)}=e-e^x$

$\Rightarrow\quad f(x)=\log_e(e-e^x)$

Domain $e-e^x>0$

$x<1\Rightarrow x\in(-\infty,1)$

Range $(-\infty,1)$

38. Gives $y+|y|=x+|x|$

If $x>0, y>0\Rightarrow 2y=2x\Rightarrow y=x$

$x<0, y>0\Rightarrow 2y=0 \quad\Rightarrow y=0$

$x>0, y<0\Rightarrow 0=2x \quad\Rightarrow x=0$

$x<0, y<0\Rightarrow 0=0 \quad\Rightarrow$ whole region of III quadrant.

For person to be safe there should not be point common to the given curves and the voltage field graph. Only $y=m+|x|$ does not have any point of intersection with the curve.

39. Gives $|f(x)+6-x^2|=|f(x)|+|4-x^2|+2$

$\Rightarrow\quad |f(x)+2+(4-x^2)|=|f(x)|+|4-x^2|+2$

Since $|a+b+c| = |a|+|b|+|c|$

If $a \geq 0, b \geq 0, c \geq 0$ or $a \leq 0, b \leq 0, c \leq 0$

$\Rightarrow$ $f(x) \geq 0$ and $4 - x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$ and $f(x) \geq 0$

40. $f(x) = \cos px + \sin x$

Period : L.C.M. of $\left(\frac{2\pi}{p}, \frac{2\pi}{1}\right)$

For period to exist p should be a rational number.

41. $y = f(e^x) + f(\ln|x|)$

Domain $f(x) = (0, 1)$

$\Rightarrow$ $0 < e^x < 1 \Rightarrow x < 0$ ...(1)

and $0 < \ln|x| < 1 \Rightarrow 1 < |x| < e \Rightarrow x \in (-e, -1) \cup (1, e)$ ...(2)

Taking intersection $x \in (-e, -1)$

42. Givens $f(1) = 2,\ f(2) = 3,\ f(3) = 4,\ f(4) = 1,\ g(1) = 3$ and $f[g(x)] = g[f(x)]$

at $x = 1$, $f[g(1)] = g[f(1)] \Rightarrow f(3) = g(2) \Rightarrow g(2) = 4$

at $x = 2$ $f[g(2)] = g[f(2)] \Rightarrow f(4) = g(3) \Rightarrow g(3) = 1$

at $x = 3$ $f[g(3)] = g[f(3)] \Rightarrow f(1) = g(4) \Rightarrow g(4) = 2$

43. Gives $[y + [y]] = 2\cos x \Rightarrow [y] + [y] = 2\cos x \Rightarrow 2[y] = 2\cos x;\ [y] = \cos x$ ...(1)

where
$$y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$$
$$y = \frac{1}{3}[\sin x + [\sin x] + [\sin x]]$$
$$y = \frac{1}{3}(3[\sin x])$$
$$y = [\sin x] \quad \text{...(2)}$$

From eqn. (1) & (2),
$$[\sin x] = \cos x$$
$$\Rightarrow \cos x = 0, 1, -1$$

Hence, no solution.

44. $f(x) = \dfrac{x^{2n}}{(x^{2n}\operatorname{sgn} x)^{2n+1}} \left[\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right]$ $x \neq 0$ and $f(0) = 1$

when $f(x) = \dfrac{(x^{2n})}{(x^{2n})^{2n+1}} \left[\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right]$; $x > 0$

$$f(x) = \frac{x^{2n}}{-(x^{2n})^{2n+1}}\left[\frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right]; \ x<0$$

Clearly, $f(x) = f(-x)$. Hence, $f(x)$ is even function.

45. $f(n) = 2(f(1) + f(2) \ldots\ldots + f(n-1))$

$f(2) = 2f(1)$

$$f(3) = 2[f(1) + f(2)] = 2\left[\frac{f(2)}{2} + f(2)\right] = 3f(2)$$

$$f(4) = 2[f(1) + f(2) + f(3)] = 2\left[\frac{f(3)}{2} + f(3)\right] = 3f(3) = 3^2 f(2)$$

$\vdots$

$$\sum_{r=1}^{m} f(r) = f(1) + f(2) \ldots\ldots + f(m) = f(1) + f(2) + 3f(2) \ldots\ldots +$$

$$= f(1) + f(2)[1 + 3 + 3^2 \ldots\ldots 3^{m-2}]$$

$$= f(1) + 2\cdot\frac{(3^{m-1} - 1)}{(3-1)} = 3^{m-1}$$

46. Gives
$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

$$f(f(x)) = \frac{x}{\sqrt{1+2x^2}}$$

$$f(f(f(x))) = \frac{x}{\sqrt{1+3x^2}}$$

$\vdots$

$$\underbrace{fofo\ldots\ldots fof(x)}_{n \text{ times}} = \frac{x}{\sqrt{1+nx^2}} = \frac{x}{\sqrt{1+\left(\sum_{r=1}^{n} 1\right)x^2}}$$

47. $f(x) = 2x + |\cos x|$

Range $f(x) = R$ = codomain $\Rightarrow$ onto.

Clearly, $f(x)$ is increasing function $\Rightarrow$ one-one function.

48. **Gives $f(x) = x^3 + x^2 + 3x + \sin x$**

Since, $f(x)$ is continuous function.

and $f(x) = \infty$ as $x \to \infty$

$f(x) = -\infty$ as $x \to -\infty$

Range $f(x) = R =$ codomains $\Rightarrow$ onto function

and $f'(x) = 3x^2 + 2x + 3 + \cos x = 3\left(x + \frac{1}{3}\right)^2 + \frac{8}{3} + \cos x \Rightarrow f'(x) > 0$

Hence, $f(x)$ is one-one.

49. $f(x) = \{x\} + \{x+1\} + \ldots \{x+99\}$

Since $\{x\} = \{x + I\}$ where $I =$ integer

$f(x) = \underbrace{\{x\} + \{x\} \ldots \{x\}}_{100 \text{ times}}$

$f(x) = 100\{x\} \Rightarrow f(\sqrt{2}) = 100\{\sqrt{2}\} = 100 \times 0.414 = 41.4$

$[f(\sqrt{2})] = 41$

50. $|\cot x + \operatorname{cosec} x| = |\cot x| + |\operatorname{cosec} x|;\ x \in [0, 2\pi]$

$\Rightarrow$ $\cot x \geq 0$ and $\operatorname{cosec} x \geq 0 \Rightarrow 1^{\text{st}}$ quadrant

or $\cot x \leq 0$ and $\operatorname{cosec} x \leq 0 \Rightarrow 4^{\text{th}}$ quadrant

Hence, $x \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right)$

51. If $f(4+x) = f(4-x)$

$\Rightarrow$ $f(x)$ is symmetric about $x = 4$.

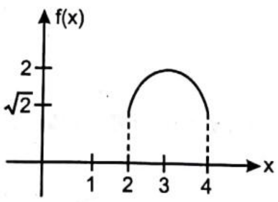
Roots of $f(x) = 0$ are of the form

$4-\alpha, 4+\alpha, 4-\beta, 4+\beta, 4-\gamma, 4+\gamma, 4-\delta, 4+\delta$

52. $f(x) + x - 6 = (x-1)(x-2)(x-3)(x-4)(x-5)$

$\Rightarrow$ $f(6) = 120$

53. $f(x) = \sqrt{x-2} + \sqrt{4-x}$



54. $\left[\frac{x}{9}\right] = \left[\frac{x}{11}\right]$

$x \in [1, 9) \cup [11, 18) \cup [22, 27) \cup [33, 36) \cup [44, 45)$

55. $\log_{\left[x+\frac{1}{2}\right]}(2x^2 + x - 1)$

$\left[x + \frac{1}{2}\right] > 0,\ \left[x + \frac{1}{2}\right] \neq 1\ \&\ 2x^2 + x - 1 > 0$

$\Rightarrow\ x+\frac{1}{2}\ge 2$ & $(2x-1)(x+1)>0$

$x\ge\frac{3}{2}$ & $x(-\infty,-1)\cup\left(\frac{1}{2},\infty\right)$

$\Rightarrow\ x\in\left[\frac{3}{2},\infty\right)$

56. $[x^2]+[x]-2=0$

Let $[x]=t$

$\Rightarrow\ t^2+t-2=0$

$\Rightarrow\ (t+2)(t-1)=0$

$\Rightarrow\ t=-2$ or $t=1$

$\Rightarrow\ [x]=-2$ or $[x]=1$

$\Rightarrow\ x\in[-2,-1)\cup[1,2)$

58. $f(x)$ is many one function.

59. $f(f(x))=2+f(x)\quad f(x)\ge 0$

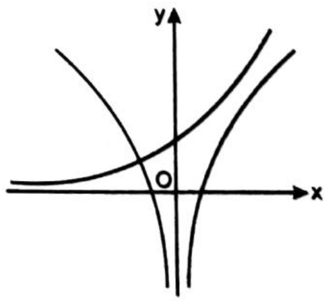
$=2-f(x)\quad f(x)<0$

$f(f(x))=4+x\quad x\ge 0$

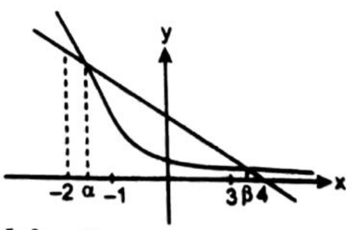
$=4-(x)\quad x<0$

60. $f'(x)=\frac{7(3x^2-2x+3)}{(3+3x-4x^2)^2}>0\Rightarrow f(x)\uparrow$

61.



62.



$[\alpha]=-2$

$[\beta]=3$

63. $f(x)=\sqrt{\sin(\log_7(\cos(\sin x)))}$

$\cos(\sin x)\le 1 \Rightarrow \cos(\sin x)=1 \Rightarrow f(x)=0$

64. $-3\le|[x]|\le 2 \Rightarrow -2\le[x]\le 2 \Rightarrow -2\le x<3$

65. $f(x)=\frac{\pi}{2}+\cot^{-1}\{-x\}$

$0\le\{-x\}<1 \Rightarrow \frac{\pi}{4}<\cot^{-1}\{-x\}\le\frac{\pi}{2}$

66. $f(f(x))=x$

$f_{2008}(x)+f_{2009}(x)=x+f(x)=x+\frac{3x+5}{2x-3}=\frac{2x^2+5}{2x-3}$

67. $f(x)=\left(x+\frac{1}{x}+1\right)\left(x^2+\frac{1}{x^2}\right)$; $x^2+\frac{1}{x^2}\ge 2$; $x+\frac{1}{x}+1\ge 3 \Rightarrow f(x)\ge 6$

68. $f(x)=e^{x^3-3x^2-9x+2}$

$f'(x)=e^{(x^3-3x^2-9x+2)}3(x-3)(x+1)$

$\Rightarrow$ $f(x)$ is many one.

at $x=-1$, $f(x)=e^7$

at $x\to-\infty$, $f(x)\to 0$

Range of $f(x)$ is $(0,e^7]$.

69. $D_f:(-2,1)$

$-\infty<\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)<\infty$

$-1\le\sin\left(\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)\le 1$

70. $f'(x)\ge 0\ \forall\ x\in R \Rightarrow 3x^2+2(a+2)x+3a\ge 0\ \forall\ x\in R$

$\Rightarrow D\le 0$

$\Rightarrow 4(a+2)^2-4\cdot 9a\le 0$

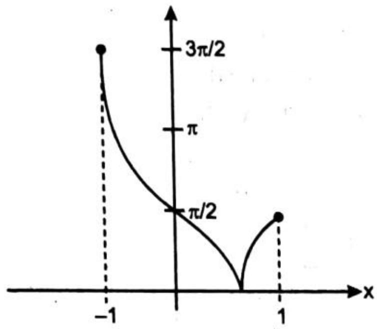
$\Rightarrow a^2-5a+4\le 0 \Rightarrow (a-1)(a-4)\le 0$

$\Rightarrow a\in[1,4]$

71. Min. value of $3x^2+bx+c=0$

$\Rightarrow$ $D=0$

72. $f(x)=\sin^{-1}x-\cos^{-1}x=2\sin^{-1}x-\frac{\pi}{2}$

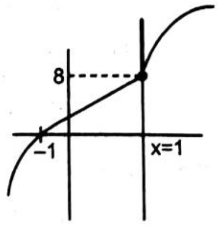


73. is one-one when

$$2^3 = \ln 1 + b^2 - 3b + 10$$

$\Rightarrow \quad b^2 - 3b + 2 = 0$

$\Rightarrow \quad b = 1, 2$



80. We have, $[x]^2 - 7[x] + 10 < 0$

$\Rightarrow \quad ([x]-5)([x]-2) < 0$

$\Rightarrow \quad 2 < [x] < 5$

$\Rightarrow \quad [x] = 3 \text{ or } 4$

$\Rightarrow \quad x \in [3, 5)$

and $\quad 4[y]^2 - 16[y] + 7 < 0$

$\quad (2[y]-7)(2[y]-1) < 0$

$\Rightarrow \quad \frac{1}{2} < [y] < \frac{7}{2}$

$\Rightarrow \quad [y] = 1 \text{ or } 2 \text{ or } 3$

$\Rightarrow \quad y \in [1, 4)$

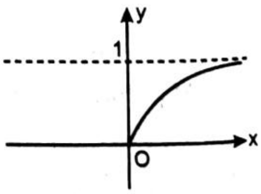
Therefore, $\quad x + y \in [4, 9)$

$\quad [x + y] \in \{4, 5, 6, 7, 8\}$

Hence, $[x + y]$ cannot be 9.

81. $f : R \to R \qquad f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$

$$f(x) = \begin{cases} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} & \text{if } x \geq 0 \\ \dfrac{e^{-x} - e^{-x}}{e^x + e^{-x}} & \text{if } x < 0 \end{cases}$$



Many one into function.

82. $f(x)$ such $f(1-x)+2f(x)=3x\ \forall\ x \in R$

$$x \to \left(\frac{1}{2}+x\right)$$

$$f\left(\frac{1}{2}-x\right)+2f\left(\frac{1}{2}+x\right)=3\left(\frac{1}{2}+x\right) \quad \ldots(1)$$

$$x \to \left(\frac{1}{2}-x\right)$$

$$f\left(\frac{1}{2}+x\right)+2f\left(\frac{1}{2}-x\right)=3\left(\frac{1}{2}-x\right) \quad \ldots(2)$$

(1) + (2)

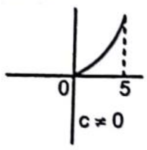
$$3\left(f\left(\frac{1}{2}+x\right)+f\left(\frac{1}{2}-x\right)\right)=3; \quad f\left(\frac{1}{2}-x\right)=1-f\left(\frac{1}{2}+x\right)$$

$$1+f\left(\frac{1}{2}+x\right)=\frac{3}{2}+3x; \quad f\left(\frac{1}{2}+x\right)=\frac{1}{2}+3x$$

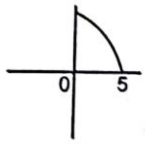
$$x=-\frac{1}{2} \Rightarrow f(0)=\frac{1}{2}-\frac{3}{2}=-1$$

83. $f:[0,5] \to [0,5]$

$f(x)=ax^2+bx+c \qquad a, b, c \in R, abc \neq 0$



or



$$25a+5b+c=0$$

$$f(5)=0$$

$ax^2+bx+c=0\,(\alpha) \qquad \frac{c}{a}=5\times\beta$

$cx^2+bx+a=0\left(\frac{1}{\alpha}\right) \qquad \beta=\frac{1}{a}$

So, roots are $\left(a, \frac{1}{5}\right)$.

84. $f(x)=x^2+\lambda x+\mu\cos x$

$f(x)=x$

85. $f(k)=\text{odd}$

$f(k+1)=\text{even} \qquad k=1,2,3$

$f(1) \Rightarrow$ odd
$f(2) \Rightarrow$ even
$f(3) \Rightarrow$ odd
$f(4) \Rightarrow$ even
Hence, 4 functions.

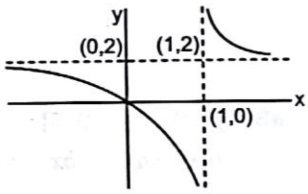
$f(1) \Rightarrow$ even
$f(2) \Rightarrow$ odd
$f(3) \Rightarrow$ even
$f(4) \Rightarrow$ odd
Hence, 4 functions.

$f(1) \Rightarrow$ odd
$f(2) \Rightarrow$ even
$f(3) \Rightarrow$ even
$f(4) \Rightarrow$ odd
Hence, 4 functions.

86. $y = \tan(\sin x)$.

Here function is continuous and differentiable and $y_{max} = \tan(1)$; $y_{min} = -\tan 1$.

87. $f(x) = \dfrac{2x}{x-1}$



$$y = 2 + \frac{2}{(x-1)}$$

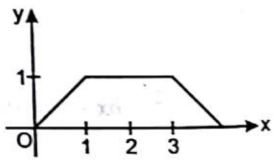
$$(y-2)(x-1) = 2$$

88. $R_f = [-2, 4]$

$R_g = [-1, 2]$

89. $f(x) = (x^4 + 1) + \dfrac{1}{x^2 + x + 1}$

90. $0 \le f(x) \le 1$



$0 \le 7f(x) \le 7$

$-1 \le \sin(7f(x)) \le 1$

91. $\ln|\ln|x|| \ge 0 \quad \cap \quad |x|^2 - 7|x| + 10 \le 0$

$|\ln|x|| \ge 1 \qquad (|x|-2)(|x|-5) \le 0$

$\ln|x| \in (-\infty, -1) \cup [1, \infty) \qquad 2 \le |x| \le 5$

$|x| \in \left(0, \frac{1}{e}\right] \cup [e, \infty) \qquad x \in (-5, -2] \cup [2, 5]$

$$x \in (-\infty, -e] \cup \left[-\frac{1}{e}, 0\right) \cup \left(0, \frac{1}{e}\right] \cup [e, \infty)$$

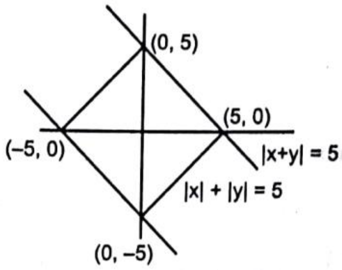
92. $\log_{[x]+3\{x\}}\left(\left([x]-\frac{5}{2}\right)^2+\frac{3}{4}\right)\geq 0 \Rightarrow [x]+3\{x\}>1$

93. $x-3=X \qquad |X|+|Y|=5$

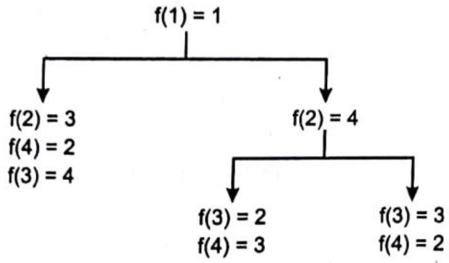
$y-1=Y$

$x+y-4=X+Y \qquad |X+Y|=5$

number of pairs of $(x, y)=12$



95.



96. $x^2-x\neq 0 \Rightarrow x\neq 0, 1$

97. Total one-one function – (at least one get right place) + (at least two get right place) – (at least three get right place) + (at least four get right place)

$$= {}^6C_4\times 4!-{}^4C_1\times{}^5C_3\times 3!+{}^4C_2\times{}^4C_2\times 2!-{}^4C_3\times{}^3C_1+{}^4C_4=181$$

98. $f(x)=x^2-2x-3$

$g(x)=f^{-1}(x)=1+\sqrt{x+4} \qquad x\geq -4$

$f(x)=g(x)=f^{-1}(x)\Rightarrow f(x)=x$

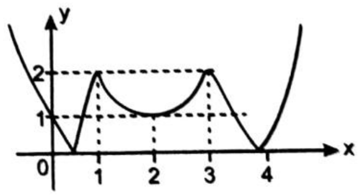
$\Rightarrow \; x^2-3x-3=0 \quad \Rightarrow x=\frac{3+\sqrt{21}}{2}$

Exercise-2 : One or More than One Answer is/are Correct

1. $f(-4) = f(4) = 40$

$f(-13) = f(13) = f(3) = 19$

$f(-11) = f(11) = f(1) = 2$

2. 

3. $f(x) = \cos^{-1}\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)$ is defined when

$\frac{x}{2} \neq (2n-1)\frac{\pi}{2}$

$\Rightarrow$ $x \neq (2n-1)\pi$

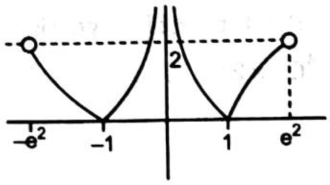
Domain $= R - \{(2n-1)\pi : n \in I\}$

$\therefore$ Range $= [0, \pi)$

$f(x) = \cos^{-1}(\cos x)$

$f(x)$ is even function.

when $x \in (\pi, 2\pi)$, then $f(x) = 2\pi - x$ is differentiable.

4. 

$0 < |k-1| - 3 < 2$

$\Rightarrow$ $k \in (-4, -2) \cup (4, 6)$

5. (a) $D_f \in R$

(b) $D_f \in R$

(c) $f(x)=\sqrt{2\cos^2 x+\cos x+\frac{1}{8}}$

$D_f \in R$

(d) $\ln(1+|x|)\geq 0$

$D_f \in \left\{\frac{(2n+1)\pi}{2}\right\}$

6. $f\left(\frac{3}{2}\right)=\frac{9}{4}$

$f\left(f\left(\frac{3}{2}\right)\right)=\frac{3}{2}$

$f\left[f\left(f\left(\frac{3}{2}\right)\right)\right]=\frac{9}{4}$

$f\left(\frac{5}{2}\right)=2$

$f\left(f\left(\frac{5}{2}\right)\right)=1$

$f\left(f\left(f\left(\frac{5}{2}\right)\right)\right)=1$

8. $f^{-1}(f(x))=f(f^{-1}(x))=x$

if $f(f^{-1}(x))=f^{-1}(x) \Rightarrow x=f^{-1}(x)$

if $f(f^{-1}(x))=f^{-1}(x) \Rightarrow f(f^{-1}(f(x)))=f^{-1}(f(x)) \Rightarrow f(x)=f^{-1}(f(x))=x$

9. $f(x)=\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3}\cdot\sqrt{1-x^2}}{2}\right)$

Let $x=\cos\theta$

$f(x)=\cos^{-1}(\cos\theta)+\cos^{-1}\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)$

$=\cos^{-1}(\cos\theta)+\cos^{-1}\left(\cos\left(\theta-\frac{\pi}{3}\right)\right)$

$=\frac{\pi}{3} \qquad 0\leq\theta\leq\frac{\pi}{3}$

$=2\theta-\frac{\pi}{3} \qquad \frac{\pi}{3}<\theta\leq\pi$

10. $f(x) = \cos^{-1}(-\{-x\})$

$-\{-x\} \in (-1, 0] \Rightarrow \cos^{-1}(-\{-x\}) \in \left[\frac{\pi}{2}, \pi\right)$

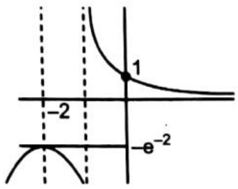
12. $h(x) = [\ln x - 1] + [1 - \ln x]$

$\Rightarrow \quad h(x) = \begin{cases} -1, & \ln x - 1 \notin I \\ 0, & \ln x - 1 \in I \end{cases}$

14. $f(x) = \frac{1}{2}$, $f(x)$ is periodic & constant function.

16. $f(x) = \frac{e^{-x}}{1+x}$

$f'(x) = \frac{-e^{-x}(x+2)}{(1+x)^2}$



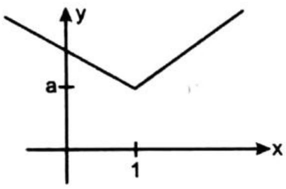
17. $[x] = \frac{2x\{x\}}{x + \{x\}} = \frac{2\{x\}([x] + \{x\})}{[x] + 2\{x\}}$

$\Rightarrow \qquad [x]^2 = 2\{x\}^2$

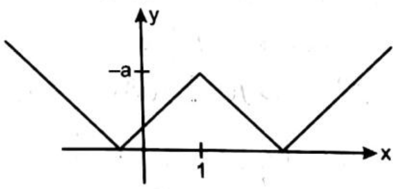
$\Rightarrow \qquad x = 1 + \frac{1}{\sqrt{2}} \qquad\qquad (\because x \in R^+)$

18. $||x - 1| + a| = 4$

if $a > 0$



if $a < 0$



(a) if eq. has three distinct real root then $a < 0$ and $a = -4$

(b) 4 distinct roots for $a \in (-\infty, -4)$

(c) if $-4 < a < 4$, there are two distinct real roots

(d) if $a > 4$, no real root.

19. (a) $f_2(x) = (\sin x)^{1/2} + (\cos x)^{1/2}$

$\sqrt{\sin x} > \sin^2 x$; $\sqrt{\cos x} > \cos^2 x \Rightarrow \sqrt{\sin x} + \sqrt{\cos x} > 1$

(b) $f_2(x) = (\sin x)^{1/2} + (\cos x)^{1/2} \qquad \Rightarrow f_2(x) = 1$ at $x = 2k\pi$

(c) $f_2(x) = (\sin x)^{1/2} + (\cos x)^{1/2}$; $f_3(x) = (\sin x)^{1/3} + (\cos x)^{1/3}$

if $x \in (2k\pi, 2k\pi + \pi/2) \qquad 0 < \sin x < 1$ and $0 < \cos x < 1$

As power increases, value of function decreases.

$\Rightarrow \quad f_2(x) < f_3(x)$

(d) $f_3(x) = (\sin x)^{1/3} + (\cos x)^{1/3}$

$f_5(x) = (\sin x)^{1/5} + (\cos x)^{1/5}$

$\Rightarrow \quad f_3(x) < f_5(x)$

20. $-1 \le \log_3\left(\frac{x^2}{3}\right) \le 1 \Rightarrow \frac{1}{3} \le \frac{x^2}{3} \le 3$

Range is $[0, 1]$.

21. $\frac{3x-1}{2} = n$

$\left[\frac{4n+5}{9}\right] + \left[\frac{4n+5}{9} + \frac{1}{2}\right] = n$

$$\left.\begin{matrix} n = 2 \\ \vdots \\ \vdots \\ n = 10 \end{matrix}\right\}$$

22. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

$$= 1 - \frac{3}{4}\left(\frac{1-\cos 4\theta}{2}\right) = \frac{5}{8} + \frac{3}{8}\cos(4\theta)$$

$$= \frac{5}{8} + \frac{3}{8}\cos(x)$$

23. (a) $g(f(x)) = \ln(\sin x)$

(b) $x^2 + (a-1)x + 9 > 0 \ \forall \ x \in R$

$(a-1)^2 - 36 < 0 \Rightarrow -5 < a < 7$

(c) $f(f(x)) = (2011 - (2011 - x^{2012}))^{1/2012} = x$

24. $\left[\frac{1}{4} + \frac{150}{200}\right] + \left[\frac{1}{4} + \frac{151}{200}\right] + \ldots + \left[\frac{1}{4} + \frac{199}{200}\right] = 50$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 4 to 6

Sol. $f(x) = \sqrt{\theta x^2 - 2(\theta^2 - 3)x - 12\theta}$

$g(x) = \ln(x^2 - 49)$

if domain of $f + g$ is same as domain of g. Then

$$\theta x^2 - 2(\theta^2 - 3)x - 12\theta \ge 0 \ \forall \ x \in (-\infty, -7) \cup (7, \infty)$$

$\Rightarrow \qquad \theta \in \left[\frac{6}{7}, \frac{7}{2}\right]$

$$h(\theta) = \ln\left[\int_0^{\theta} 4\cos^2 t\, dt - \theta^2\right] = \ln[2\theta + \sin 2\theta - \theta^2]$$

Paragraph for Question Nos. 7 to 8

7. For $x \in [5^4, 5^5]$

$$f(x) = \alpha^4\left[2 - \left|\frac{x}{5^4} - 3\right|\right]$$

$\alpha = 2$

$f(x)_{\max} = 32$

8. $\alpha = 5$

$$f(x) = 5^4\left[2 - \left|\frac{x}{5^4} - 3\right|\right]$$

$$f(2007) = 5^4\left[2 - \frac{2007}{625} + 3\right] = 1118$$

Paragraph for Question Nos. 9 to 10

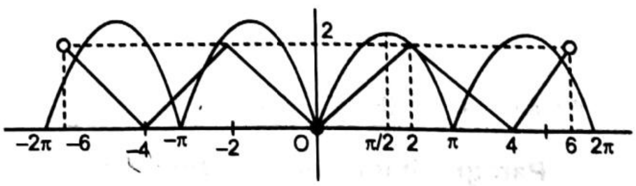
9. $f(x) \rightarrow \begin{cases} x & 0 \le x \le 2 \\ -x & -2 \le x < 0 \end{cases}$

$f(x) = f(x+4)$

$\{f(5.12)\} = \{f(1.12)\} = 0.12$

$\{f(7.88)\} = \{f(3.88)\} = \{f(-0.12)\} = 0.12$

10.



Paragraph for Question Nos. 13 to 14

13. $f(x) = 3$

$3 + \ln b_1, 3 + \ln b_2, 3 + \ln b_3$ are in A.P.

14. $y = 3x^2$

Let slope of tangent be m.

$\Rightarrow \quad y = m(x-2)$

$\Rightarrow \quad m(x_1 - 2) = 3x_1^2$

Also, $\quad m = 6x_1$

$\Rightarrow \quad 6x_1(x_1 - 2) = 3x_1^2$

$x_1 = 4$

$m = 24$

Paragraph for Question Nos. 15 to 16

15. $y = 2^{x^4 - 4x^2} \Rightarrow x^4 - 4x^2 = \log_2 y$

$$x^2 = \frac{4 + \sqrt{16 - 4\log_2 y}}{2} \Rightarrow x = \sqrt{2 + \sqrt{4 - \log_2 y}}$$

16. $g(x) = 1 + \dfrac{6}{\sin x - 2} \Rightarrow$ Range $[-5, -2]$

Exercise-4 : Matching Type Problems

1.
$[x] + \{x\} + [y] + \{z\} = 12.7$ ...(i)

$[x] + \{y\} + [z] + \{z\} = 4.1$ ...(ii)

$\{x\} + [y] + \{y\} + [z] = 2$ ...(iii)

Adding (i), (ii) & (iii),

$\Rightarrow \quad [x] + \{x\} + [y] + \{y\} + [z] + \{z\} = 9.4$

$\Rightarrow \quad \{y\} + [z] = -3.3,\ \{x\} + [y] = 5.3,\ [x] + \{z\} = 7.4$

$\Rightarrow \quad \{y\} = 0.7,\ [z] = -4,\ \{x\} = 0.3,\ [y] = 5$

$[x] = 7,\ \{z\} = 0.4$

4. (A) $f(x) = \sin^2 2x - 2\sin^2 x = 2\sin^2 x \cos 2x$

Function is even, hence many one, function is also periodic.

$$f(x) = (1 - \cos 2x)\cos 2x = \frac{1}{4} - \left(\cos 2x - \frac{1}{2}\right)^2$$

Range of function is $\left[-2, \frac{1}{4}\right]$.

(B) $f(x) = 4x$

(C) $f(x) = \sqrt{\ln(\cos(\sin x))}$

$\ln(\cos(\sin x)) \geq 0$

$\Rightarrow \quad \cos(\sin x) = 1$

$\Rightarrow \quad f(x) = 0$

(D) $f(x) = \tan^{-1}\left(\dfrac{x^2+1}{x^2+\sqrt{3}}\right)$

$f(x)$ is even & hence many one.

Range is $\left[\dfrac{\pi}{6}, \dfrac{\pi}{4}\right)$.

7. (A) Domain of $g(x)$ is $[0, 3]$.

(B) Range of $g(x)$ is $[0, 3]$.

(C) $f(f(f(2))) = 1$

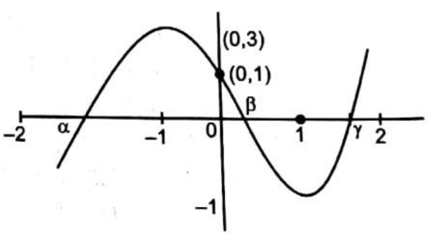
$f(f(f(3))) = 2$

(D) $m = 3$

Exercise-5 : Subjective Type Problems

1. $f(x) - 2x + 1 = (x-1)(x-2)(x-3)(x-4)(x-5)(2009x - \alpha)$

2. $f(x) = x^3 - 3x + 1$



$f(f(x)) = 0$

Let $f(x) = t$

$\Rightarrow \quad f(t) = 0$

$\Rightarrow \quad t = \alpha, \beta, \gamma$

$\Rightarrow \quad f(x) = \alpha,\ \alpha \in (-2, -1)$

No. of solution = 1

$f(x) = \beta,\ \beta \in (0, 1)$

No. of solution = 3

$f(x) = \gamma,\ \gamma \in (1, 2)$

No. of solution = 3

3. Put $x = y = 0$ $\quad f(1) = 4$

Put $x = 0, y = 1$ $\quad f(2) = 9$

4. $-1 \le \dfrac{2x}{3} \le 1 \quad \Rightarrow \dfrac{-3}{2} \le x \le \dfrac{3}{2}$

$12 - 3^x - \dfrac{27}{3^x} \ge 0 \Rightarrow (3^x - 3)(3^x - 9) \le 0 \Rightarrow 1 \le x \le 2$

5. $\sin^{-1}(0) + \cos^{-1}(-1) = \pi \qquad 0 \le x^2 < \dfrac{4}{9}$

$\sin^{-1}(1) + \cos^{-1}(0) = \pi \qquad \dfrac{4}{9} \le x^2 < \dfrac{13}{9}$

8. Let $P(x) = ax^4 + bx^3 + cx^2 + dx + 2$

$P(1) = a + b + c + d + 2 = 5$ ...(1)

$P(-1) = a - b + c - d + 2 = 5$ ...(2)

$\Rightarrow$ $b + d = 0$ and $a + c = 3$

$P(2) = 16a + 8b + 4c + 2d + 2 = 2$ ...(3)

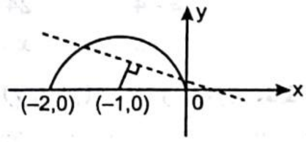
$P(-2) = 16a - 8b + 4c - 2d + 2 = 2$ ...(4)

$\Rightarrow$ $4a + c = 0$ and $4b + d = 0$

$\Rightarrow$ $b = d = 0$ and $a = -1, c = 4$

$\Rightarrow$ $P(x) = -x^4 + 4x^2 + 2$

9. $(x+1)^2 + y^2 = 1$ $(\because y > 0)$



$x + y = k$

$\left|\frac{k+1}{\sqrt{2}}\right| < 1$

$-\sqrt{2} - 1 < k < \sqrt{2} - 1$

$\Rightarrow$ $0 < k < \sqrt{2} - 1$ $(\because k > 0)$

10. $\sqrt{[x] + \left[\frac{x}{2}\right]} + \left[\frac{x}{3}\right] = 3$

$\therefore$ when $\left[\frac{x}{3}\right]$ is an integer then definitely $\sqrt{[x] + \left[\frac{x}{2}\right]}$ is also an integer.

So, $\sqrt{[x] + \left[\frac{x}{2}\right]} = 2$ and $\left[\frac{x}{3}\right] = 1$ (and check like this)

$[x] + \left[\frac{x}{2}\right] = 4,\ \left[\frac{x}{3}\right] = 1 \Rightarrow x \in [3, 6)$

when $x \in [3, 4)$

$[x] = 3, \left[\frac{x}{2}\right] = 1$

So, $x \in [3, 4)$ satisfies.

when $x \in [4, 5)$ $[x] = 4$ $\left[\frac{x}{2}\right] = 2 \Rightarrow [x] + \left[\frac{x}{2}\right] = 6 \neq 4$ not satisfies, similarly on checking all possibilities we have only $x \in [3, 4)$.

$\therefore$ $a = 3, b = 4$

11. $f(f(x)) = \dfrac{1}{\sqrt[2011]{1 - \dfrac{1}{1 - x^{2011}}}} = \dfrac{\sqrt[2011]{1 - x^{2011}}}{-x}$

$$f(f(f(x))) = \frac{\sqrt[2011]{\left(1-\frac{-1}{1-x^{2011}}\right)}}{\frac{-1}{\sqrt[2011]{1-x^{2011}}}} = \frac{\frac{-x}{\sqrt[2011]{1-x^{2011}}}}{\frac{-1}{\sqrt[2011]{1-x^{2011}}}} = x$$

$\therefore \quad f_{2013}(x) = x = \{-x\}$

12. $f(x) = 0 \qquad 0 < x < 6$

$= -1 \qquad 6 \le x < 12$

$= -2 \qquad 12 \le x < 18$

$= -3 \qquad 18 \le x < 24$

$= -4 \qquad 24 \le x < 30$

$= -5 \qquad x = 30$

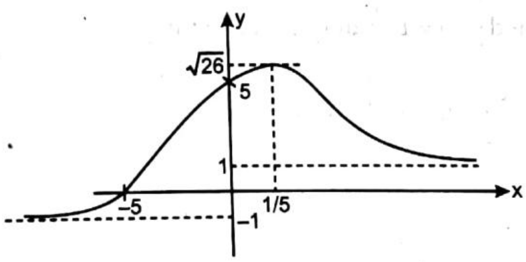
13. $(f(x, y))^2 - (g(x, y))^2 = \frac{1}{2}$

$f(x, y) \cdot g(x, y) = \frac{\sqrt{3}}{4}$

$\Rightarrow \quad f(x, y) = x^2 - y^2 = \pm\frac{\sqrt{3}}{2}$

$g(x, y) = 2xy = \pm\frac{1}{2}$

14. $f(x) = \frac{x+5}{\sqrt{x^2+1}}$



15. $f(x)$ is injective for $x \in \left(-\infty, \frac{1}{5}\right]$

$[\lambda] = \left[\frac{1}{5}\right] = 0$

16. $f: R \to R \; f(x) = \frac{x^3}{3} + (m-1)x^2 + (m+5)x + n$

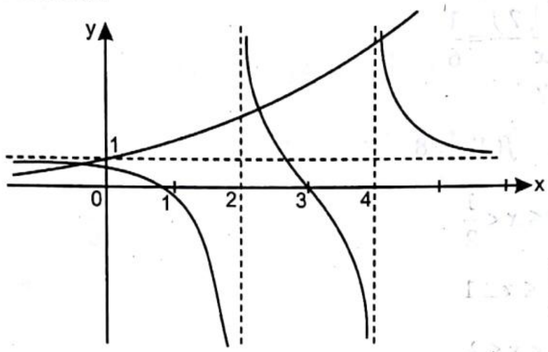
$f'(x) = x^2 + 2(m-1)x + (m+5) \ge 0$

$\Delta \le 0$

$$4(m-1)^2 - 4(m+5) \le 0$$
$$m^2 - 3m - 4 \le 0$$
$$(m-4)(m+1) \le 0$$
$$-1 \le m \le 4$$

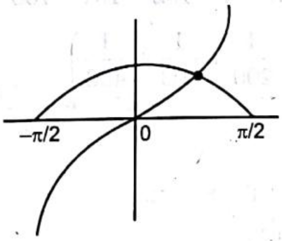
17. $f(x) = \dfrac{(x-1)(x-3)}{(x-2)(x-4)} - e^x$

$f(x) = 0$ has three solutions.



$f(-x) = \dfrac{(x+1)(x+3)}{(x+2)(x+4)} - e^{-x} = 0$ has three solutions.

$x^3 = \cos x$

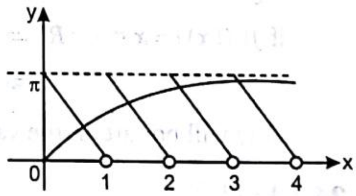


one solution

there are total 7 solutions.

18: $\cos^{-1}\left(\dfrac{2}{(1+x)^2} - 1\right) = \pi(1 - \{x\})$

there are total 76 solutions.



19. $f(x) = x^2 - bx + c = 0 \begin{smallmatrix} \nearrow p_1 \\ \searrow p_2 \end{smallmatrix}$

$p_1 + p_2 = b$ (odd no.)

$\Rightarrow \quad p_1 = 2$

$$p_1p_2 = c$$
$$b + c = (p_2 + 2) + 2p_2 = 35$$
$$\Rightarrow \quad p_2 = 11$$
$$\Rightarrow f(x) = x^2 - 13x + 22$$
$$\lambda = f(x)_{\min} = -\frac{81}{4}$$

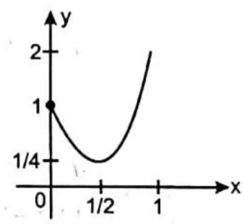
20. $f'(x) = \lim_{x\to 0} \dfrac{f(x) - f\left(\frac{x}{7}\right)}{x - \frac{x}{7}} = \dfrac{1}{6}$

$$\Rightarrow f(x) = \frac{x}{6} + 1 \Rightarrow f(42) = 8$$

21. $g(x) = f(x) \qquad 0 \le x < \frac{1}{2}$

$= \frac{1}{4} \qquad \frac{1}{2} \le x \le 1$

$= 3 - x \qquad 1 < x \le 2$



22. $x = \dfrac{10}{4}\sum_{r=3}^{100}\left(\dfrac{1}{r-2} - \dfrac{1}{r+2}\right) = \dfrac{10}{4}\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{102} - \dfrac{1}{101} - \dfrac{1}{100} - \dfrac{1}{99}\right)$

$$= 5 \times 49\left(\frac{1}{99} + \frac{1}{200} + \frac{1}{303} + \frac{1}{408}\right)$$

$$[x] = 5$$

23. $f(x) = x$ has two real roots.

$cx^2 + (d-a)x - b = 0$ with roots 7 and 11

$\dfrac{a-d}{c} = 18$ and $\dfrac{-b}{c} = 77$

if $f(f(x)) = x\ \forall\ x \in R \Rightarrow (ac + cd)x^2 + (d^2 - a^2)x - (a+d)b = 0$

$$\Rightarrow a + d = 0 \Rightarrow a = -d$$

$f(x)$ will not attain the value $\dfrac{a}{c} = 9$.

24. $A = (1, 3)$

$p \le -2^{1-1}, p \le -2^{1-3}$

$1 - 2(p+7) + 5 \le 0$ and $9 - 6(p+7) + 5 \le 0 \Rightarrow p \in [-4, -1]$

25. $y = \dfrac{x - \dfrac{1}{x}}{x^3 - \dfrac{1}{x^3} + 2}$ Let $t = x - \dfrac{1}{x} > 0$ for $x > 1$

$y = \dfrac{t}{t(t^2+3)+2}$ $x^3 - \dfrac{1}{x^3} = t(t^2+3)$

$= \dfrac{t}{t^3 + 3t + 2}$

$= \dfrac{1}{t^2 + \dfrac{2}{t} + 3}$ $\left(t^2 + \dfrac{2}{t} = t^2 + \dfrac{1}{t} + \dfrac{1}{t} \geq 3 \right.$ $\left. \therefore t^2 + \dfrac{2}{t} + 3 \geq 6 \text{ (AM} \geq \text{GM)} \right.$

$y_{\max} = \dfrac{1}{\left(t^2 + \dfrac{2}{t} + 3\right)_{\min}} = \dfrac{1}{6}$

$p = 1, q = 6$

28. $a + ar + ar^2 = 1$

$a^2r + a^2r^2 + a^2r^3 = \beta = ar(a + ar + ar^2) = ar$

$a^3r^3 = -\gamma$

29. $m = {}^6C_4 \times 1 = 15$

$n = \dfrac{6!}{3!1!1!1!3!} \times 4! + \dfrac{6!}{(2!)^4} \times 4! = 1560$

30. $\sum_{r=1}^{n} [\log_2 r] = 0 + 1 + 1 + (2+2+2+2) + \underbrace{(3+3+\ldots+3)}_{8 \text{ times}} + \ldots$

$= 2 \cdot 1 + 4 \cdot 2 + 8 \cdot 3 + \ldots +$

32. 3

$|(x-2y)(y+x)(x+3y)| = f(x,y)$

No rain, then $f(x,y) = 0$ hence 3 lines.

33. Cubic $= (x^2 - 5x + 6)(x + \alpha) + 2(Bx + 100 - 4\alpha)$

$(x^2 - 5x + 4)(x + \alpha) + Bx + 100 - 4\alpha$

Both identical $B = -2$

$\alpha = 50$

Cubic $= (x^2 - 5x + 6)(x + 50) - 4x - 200$

34. $f(\theta)=0 \Rightarrow \theta=-5\pm\sqrt{5}$

$\Rightarrow f(f(f(x)))=-5\pm\sqrt{5}$

Since $f(x)=(x+5)^2-5$

$f(f(f(x)))=-5\pm\sqrt{5}$

$((f(f(f(x))))+5)^2=-5\pm\sqrt{5}$

$(f(f)+5)^2=\sqrt{5}$

$f(f)+5=\pm 5^{1/4}$

$f(f)=-5\pm 5^{1/4}$

$(f+5)^2-5=-5\pm 5^{1/4}$

$(f+5)^2=5^{1/4}$

$f+5=\pm 5^{1/8}$

35. Let $\ln x = t$

$$y=\frac{2t^2+3t+3}{t^2+2t+2} \Rightarrow (y-2)t^2+(2y-3)t+(2y-3)>0$$

$$D\ge 0 \Rightarrow (2y-3)(2y-5)\le 0 \Rightarrow \frac{3}{2}\le y<\frac{5}{2}$$

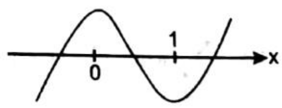
36. $P(x)=(x-3)Q_1(x)+6 \Rightarrow P(3)=6$ ...(1)

$P(x)=(x^2-9)Q(x)+(ax+b)$

$P(3)=3a+b=6$

If equation of odd degree polynomial, then $b=0, a=2$.

37. $f(x)=2x^3-3x^2+P$



$f'(x)=6x^2-6x=6x(x-1)$

$f(0)\ge 0 \cap f(1)\le 0$

$\Rightarrow P\ge 0 \cap P-1\le 0$

38. $f(x)=\dfrac{1}{\sqrt{\ln(\cos^{-1}x)}}$

$\ln(\cos^{-1}x)>0 \Rightarrow \cos^{-1}x>1$

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Chapter 2 – Limit

2 LIMIT

Exercise-1 : Single Choice Problems

1. $\lim\limits_{x\to 0} \dfrac{2\sin\left(\dfrac{x-\tan x}{2}\right)\sin\left(\dfrac{\tan x + x}{2}\right)}{\left(\dfrac{x-\tan x}{2}\right)\left(\dfrac{\tan x + x}{2}\right)} \times \left(\dfrac{x-\tan x}{x^3}\right)\left(\dfrac{x+\tan x}{x}\right)\times\dfrac{1}{4}$

$$=\frac{1}{2}\times\left(-\frac{1}{3}\right)\cdot 2=-\frac{1}{3}$$ (use expansions)

3. $a=\lim\limits_{x\to 0}\left(\dfrac{\ln(1+\cos 2x-1)}{\cos 2x-1}\right)\dfrac{(\cos 2x-1)}{3x^2}=-\dfrac{2}{3}$

$b=\lim\limits_{x\to 0}\left(\dfrac{\sin^2 2x}{4x^2}\right)\dfrac{4x^2}{x^2\left(\dfrac{1-e^x}{x}\right)}=-4$

$c=\lim\limits_{x\to 1}\dfrac{\sqrt{x}(1-x)}{\left(\dfrac{\ln(1+x-1)}{x-1}\right)(x-1)(\sqrt{x}+1)}=\dfrac{-1}{2}$

4. $f(x)=\dfrac{\pi}{2}-3\tan^{-1}x$

$g(x)=2\tan^{-1}x$

$\lim\limits_{x\to 0}\dfrac{f(x)-f(a)}{g(x)-g(a)}=\dfrac{f'(a)}{g'(a)}=-\dfrac{3}{2}$

5. $\lim\limits_{x\to 0}\left(e^{\frac{2}{x}\ln(1+x)-2}\right)^{\frac{4}{\sin x}} = e^{\lim\limits_{x\to 0}\frac{4}{\sin x}\left(e^{2\left(\frac{\ln(1+x)}{x}-1\right)}-1\right)}$

$$= e^{\lim\limits_{x\to 0}\frac{4}{\sin x}\left(\frac{e^{2\left[\frac{\ln(1+x)}{x}-1\right]}-1}{2\left(\frac{\ln(1+x)}{x}-1\right)}\right)\times 2\left[\frac{\ln(1+x)}{x}-1\right]}$$

$$= e^{\lim\limits_{x\to 0} 8\left(\frac{x-\frac{x^2}{2}+\ldots}{x}-1\right)\times\frac{1}{\sin x}}$$

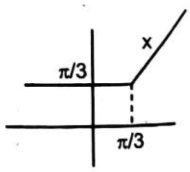
$$= e^{-\frac{8}{2}} = e^{-4}$$

6. $\lim\limits_{x\to\infty}\frac{3}{x}\left(\frac{x}{4}-\left\{\frac{x}{4}\right\}\right)=\frac{3}{4}-0=\frac{3}{4}$

$\Rightarrow \quad p+q=7$

7. $f(x)=\lim\limits_{n\to\infty}\frac{x\left(1+\left(\frac{\pi}{3x}\right)^n\right)}{1+\left(\frac{\pi}{3x}\right)^{n-1}}=x;\ x>\frac{\pi}{3}$

$$=\lim\limits_{n\to\infty}\frac{\frac{\pi}{3}\left(\left(\frac{3x}{\pi}\right)^n+1\right)}{\left(\left(\frac{3x}{\pi}\right)^{n-1}+1\right)}=\frac{\pi}{3};\ x<\frac{\pi}{3}$$



$=\frac{\pi}{3} \qquad x=\frac{\pi}{3}$

$\therefore \quad f(x)=x;\ x\ge\frac{\pi}{3}$

$\qquad =\frac{\pi}{3};\ x<\frac{\pi}{3}$

$\therefore$ Option (d) is wrong.

8. $\lim\limits_{x\to 0}\frac{\sin(\pi-\pi\cos^2(\tan(\sin x)))}{x^2}=\lim\limits_{x\to 0}\frac{\sin[\pi\sin^2(\tan(\sin x))]}{\pi\sin^2(\tan(\sin x))}\times\pi\left(\frac{\sin(\tan(\sin x))}{x}\right)^2=\pi$

9. $\lim\limits_{x\to 3^-}f(x)=\lim\limits_{x\to 3^+}f(x)\Rightarrow\lim\limits_{x\to 3^-}\frac{(27)^{\frac{(x+3)x}{27}}-9}{3^x-27}=\lim\limits_{x\to 3^+}\lambda\frac{1-\cos(x-3)}{(x-3)^2}$

$$\Rightarrow \quad \lim_{x\to 3^-} \frac{3^2\left(3^{\frac{x^2+3x}{9}-2}-1\right)}{3^3(3^{x-3}-1)}=\frac{\lambda}{2}$$

$$\Rightarrow \quad \lim_{x\to 3}\frac{1}{3}\frac{x^2+3x-18}{9(x-3)}=\frac{\lambda}{2} \Rightarrow \frac{1}{27}\cdot 9=\frac{\lambda}{2} \Rightarrow \lambda=\frac{2}{3}$$

10. $$\lim_{x\to\frac{\pi}{3}} \frac{2\sin\left(\dfrac{\frac{\pi}{3}-x}{2}\right)\cos\left(\dfrac{\frac{\pi}{3}-x}{2}\right)}{2\left(\cos x-\cos\frac{\pi}{3}\right)}=\lim_{x\to\frac{\pi}{3}}\frac{\sin\left(\dfrac{\frac{\pi}{3}-x}{2}\right)\cos\left(\dfrac{\frac{\pi}{3}-x}{2}\right)}{2\sin\left(\dfrac{\frac{\pi}{3}-x}{2}\right)\sin\left(\dfrac{\frac{\pi}{3}+x}{2}\right)}$$

$$=\frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}$$

11. $$\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{\cos^{-1}[\sin^3 x]} \Rightarrow \frac{\sin\frac{\pi}{2}}{\cos^{-1}(0)}=\frac{1}{\pi/2}=\frac{2}{\pi}$$

13. $$\lim_{x\to I^-}\{x\}=\lim_{x\to I^-} x-[x]=1;\quad \lim_{x\to I^-}\frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}=e-2$$

16. $$\lim_{x\to\infty} x\left[x^{5c-1}\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^c-1\right]=l$$

Case-I: $5c-1>0$, then $l\to\infty$

Case-II: $5c-1<0$, then $l\to-\infty$

Since limit is finite and non-zero so $5c-1=0 \Rightarrow c=\frac{1}{5}$

$$\therefore \quad \lambda=\lim_{x\to\infty} x\left[\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^{1/5}-1\right]$$

$$=\lim_{x\to\infty} x\left[1+\left(\frac{1}{5}\right)\left(\frac{7}{x}+\frac{2}{x^5}\right)+\ldots\ldots-1\right] \quad \text{(by binomial approximation)}$$

$$=\frac{7}{5}$$

17. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}\left(\frac{\cos x - 1}{x^{n-2}} - \frac{(e^x - 1)}{x^{n-2}}\right) = 0 \Rightarrow n = 1, 2, 3$

18. 1^∞ (form) $= e^{\lim_{x\to 0} \frac{1}{1-\cos x}\left(\frac{\sin x - x}{x}\right)} = e^{2\times -1/6} = e^{-1/3}$

19. $\lim_{x\to\infty} [\sqrt{x^2 - x + 1} - (ax + b)] = 0$

So $a > 0$, on rationalizing

$$\lim_{x\to\infty}\left[\frac{(x^2 - x + 1) - [a^2x^2 + b^2 + (2ab)x]}{\sqrt{x^2 - x + 1} + ax + b}\right] = 0$$

So, $1 - a^2 = 0 \quad -1 - 2ab = 0$

$a = 1$

$\lim_{n\to\infty} \sec^2[k!\pi(-1/2)] = 1 = a$

20. $f(x + T) = f(x + 2T) = \ldots\ldots = f(x + nT) = f(x)$

$$\lim_{n\to\infty} \frac{nf(x)(1 + 2 + 3 + \ldots + n)}{f(x)(1 + 2^2 + 3^2 + \ldots + n^2)} = \lim_{n\to\infty} \frac{n\left(\frac{n(n+1)}{2}\right)}{\frac{n(n+1)(2n+1)}{6}} = \frac{3}{2}$$

21. $265\left[\lim_{h\to 0} \frac{h^2 + 3}{\left(\frac{f(1-h) - f(1)}{-h}\right)\left(\frac{\sin 5h}{h}\right)}\right] = -265 \times \frac{3}{f'(1)\cdot 5} = -\frac{53\times 3}{f'(1)}$

$= -\frac{53\times 3}{-53}$ $\quad [\because f'(1) = -53]$

$= 3$

22. $\lim_{x\to 0} \frac{\cos^2 x - 1}{\cos x \cdot x^2 \cdot (x + 1)}$

$\lim_{x\to 0} -\left(\frac{\sin^2 x}{x^2}\right)\frac{-1}{\cos x(x + 1)} = -1$

23. $f(x + y) = f(x)\cdot f(y)$

$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = f(x)\left(\lim_{h\to 0} \frac{f(h) - 1}{h}\right)$

If $f(h) = 1 + hP(h) + h^2Q(h) \Rightarrow f'(x) = f(x)\lim_{h\to 0} \frac{hP(h) + h^2Q(h)}{h} = P(0)f(x)$

24. $\lim_{x\to\frac{\pi}{2}} \dfrac{\left(1-\tan\frac{x}{2}\right)(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$

$= \lim_{x\to\frac{\pi}{2}} \dfrac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)}{(\pi-2x)^3}$

Let $x=\frac{\pi}{2}+h$

$\lim_{x\to 0} \dfrac{\tan\left(-\frac{h}{2}\right)(1-\cos h)}{(-2h)^3} = \dfrac{1}{32}$

25. $\lim_{x\to\infty}\left(\dfrac{x-3}{x+2}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{-5}{x+2}\right)} = e^{-5}$

27. $\ln c = I, \quad (I \in \text{integer})$

$\Rightarrow \quad c = e^I$

c is rational when $I=0$

28. $\lim_{x\to 0}\left(1+\dfrac{a\sin bx}{\cos x}\right)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\left[1+\frac{a\sin bx}{\cos x}-1\right]} = e^{ab}$

30. $a = \lim_{x\to 1}\left(\dfrac{x}{\ln x}-\dfrac{1}{x\ln x}\right) = \lim_{x\to 1}\dfrac{x^2-1}{x\ln x} = \lim_{x\to 1}\dfrac{x+1}{x}\cdot\dfrac{x-1}{\ln x} = 2$

$b=-4, c=1, d=-2$

32. $f(x) = x^2 \qquad -1<x<0$

$= 1 \qquad x=0$

$= \dfrac{1}{x^2} \qquad 0<x<1$

$\lim_{x\to 0^-}\{f(x)\} + \lim_{x\to 1^-}\{f(x)\} + \lim_{x\to -1^-}\{f(x)\} = 0$

33. Let $\sin^{-1} x = \theta$

$\Rightarrow \quad \lim_{\theta\to\frac{\pi}{4}^+} \dfrac{\cos^{-1}\sin 2\theta}{\sin\theta-\sin\frac{\pi}{4}} = \lim_{\theta\to\frac{\pi}{4}^+} \dfrac{2\theta-\frac{\pi}{2}}{2\sin\left(\dfrac{\theta-\frac{\pi}{4}}{2}\right)\cos\left(\dfrac{\theta+\frac{\pi}{4}}{2}\right)} = 2\sqrt{2}$

$$\lim_{\theta \to \frac{\pi}{4}^-} \frac{\cos^{-1} \sin 2\theta}{\sin\theta - \sin\frac{\pi}{4}} = -2\sqrt{2}$$

34. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sin\frac{\pi}{2k} - \sin\frac{\pi}{2(k+2)} \right) + \lim_{n\to\infty} \sum_{k=1}^{n} \left(\cos\frac{\pi}{2(k+2)} - \cos\frac{\pi}{2k} \right)$

$$= \lim_{n\to\infty} \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{6} + \sin\frac{\pi}{4} - \sin\frac{\pi}{8} + \sin\frac{\pi}{6} - \sin\frac{\pi}{10} + \ldots + \sin\frac{\pi}{2n} - \sin\frac{\pi}{2(n+2)} \right)$$

$$+ \lim_{n\to\infty} \left(\cos\frac{\pi}{6} - \cos\frac{\pi}{2} + \cos\frac{\pi}{8} - \cos\frac{\pi}{4} + \cos\frac{\pi}{10} - \cos\frac{\pi}{6} + \ldots + \cos\frac{\pi}{2(n+2)} - \cos\frac{\pi}{2n} \right)$$

$$= 1 + \frac{1}{\sqrt{2}} + 2 - \frac{1}{\sqrt{2}} = 3$$

36. $\lim_{x\to 0} \frac{(\cos x)^{\frac{1}{m} - \frac{1}{n}} - 1}{x^2} = \lim_{x\to 0} \frac{\left(1 - 2\sin^2\frac{x}{2}\right)^{\frac{1}{m} - \frac{1}{n}} - 1}{x^2}$

$$= \lim_{x\to 0} -2\left(\frac{1}{m} - \frac{1}{n}\right) \frac{\sin^2\frac{x}{2}}{x^2} = \frac{m-n}{2mn}$$

37. $\lim_{x\to 0} \frac{x + xa\cos x - b\sin x}{x^3} = 1$

Using expansion,

$$\Rightarrow \lim_{x\to 0} \frac{x + xa\left(1 - \frac{x^2}{2!}\right) - b\left(x - \frac{x^3}{3!}\right)}{x^3} \Rightarrow \lim_{x\to 0} \frac{x + ax - \frac{ax^3}{2!} - bx + \frac{bx^3}{3!}}{x^3}$$

Clearly, $1 + a - b = 0$ for limit to be finite ...(1)

$$\Rightarrow \lim_{x\to 0} \left(\frac{b}{3!} - \frac{a}{2!}\right) \frac{x^3}{x^3} = 1 \Rightarrow \frac{b}{6} - \frac{a}{2} = 1 \Rightarrow b - 3a = 6 \quad \ldots(2)$$

$\Rightarrow$ From (1) and (2), $a = -\frac{5}{2}, b = -\frac{3}{2}$

38. $\lim_{x\to 0} \frac{a\cos ax - \frac{e^x(\cos x - \sin x)}{e^x \cdot \cos x}}{\sin bx + bx\cdot\cos bx} = \frac{1}{2}$

$$\Rightarrow \lim_{x\to 0} \frac{\cos^2 x - \cos x + \sin x}{\cos x(\sin bx + bx\cos bx)} = \frac{1}{2} \qquad (\because a = 1)$$

39. $\alpha = \lim_{n\to\infty} \frac{(1^3 + 2^3 + 3^3 \ldots + n^3) - (1^2 + 2^2 \ldots + n^2)}{n^4} = \lim_{n\to\infty} \frac{\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(2n+1)(n+1)}{6}}{n^4}$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{1}{n}\right)^2-\frac{(2n+1)(n+1)}{6n^3}\right]=\frac{1}{4}$$

40. $\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4} \Rightarrow \lim_{x\to 0}\frac{2\sin\left(\frac{\sin x+x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$$\Rightarrow \lim_{x\to 0}\frac{2}{x^4}\frac{(\sin x+x)}{2}\cdot\frac{(x-\sin x)}{2} \qquad \left\{\because \frac{\sin x+x}{2}\to 0;\ \frac{x-\sin x}{2}\to 0\right\}$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{2}\left(1+\frac{\sin x}{x}\right)\left(\frac{x-\sin x}{x^3}\right)\Rightarrow\frac{1}{6}$$

42.
$$u_n=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}\ldots+\frac{n}{2^n} \qquad \ldots(1)$$

$$\frac{1}{2}u_n=\ldots+\frac{1}{2^2}+\frac{2}{2^3}\ldots+\frac{n-1}{2^n}+\frac{n}{2^{n+1}} \qquad \ldots(2)$$

Substracting equation (1) and (2),

$$\frac{u_n}{2}=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}-\frac{n}{2^{n+1}} \Rightarrow \frac{u_n}{2}=\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}-\frac{n}{2^{n+1}}$$

$$\Rightarrow u_n=2\left(1-\frac{1}{2^n}\right)-\frac{n}{2^{n+1}};\ \lim_{n\to 0}u_n=2$$

43. $e^{\lim_{x\to 0}\frac{(\cos x-1)}{\sin^2 x}}+\lim_{x\to 0}\frac{\frac{\sin 2x}{2x}\cdot 2x+6x\cdot\frac{\tan^{-1}3x}{3x}+3x^2}{\frac{\ln(1+3x+\sin^2 x)}{3x+\sin^2 x}\cdot(3x+\sin^2 x)+xe^x}=\frac{1}{\sqrt{e}}+2$

44. $\tan\frac{x}{2}(1+\sec x)=\tan x$

$$f_n(x)=\tan\frac{x}{2}(1+\sec x)(1+\sec 2x)\ldots(1+\sec 2^n x)=\tan 2^n x$$

45. $\lim_{x\to\frac{\pi}{4}}(1+[x])^{\frac{1}{\ln(\tan x)}}=\lim_{x\to\frac{\pi}{4}}(1)^{\frac{1}{\ln(\tan x)}}=1$

46. $\lim_{x\to 0}\frac{\{(a-n)nx-\tan x\}}{x^2}\frac{\sin nx}{nx}xn=0$

$$\Rightarrow \{(a-n)n-1\}n=0 \Rightarrow a=n+\frac{1}{n}$$

47. $y = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{3n^3+4}{4n^4-1}}$

$$\ln y = \lim_{n\to\infty}\frac{3n^3+4}{4n^4-1}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right) = \frac{3}{4}\int_0^1 \ln x\,dx = \frac{-3}{4} \Rightarrow y = e^{-3/4}$$

48. $\lim_{x\to\infty}\frac{ax^2+bx+c}{dx+e} = \lim_{x\to\infty}\frac{ax+b+(c/x)}{d+(e/x)} = \lim_{x\to\infty}\left(\frac{a}{d}x+\frac{b}{d}\right)$

$= +\infty$ if $\left(\frac{a}{d}\right)$ is positive.

$= -\infty$ if $\left(\frac{a}{d}\right)$ is negative.

Alternate solution :

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx+e} = \lim_{x\to\infty}\frac{a+(b/x)+(c/x^2)}{(d/x)+(e/x^2)}$$

Here $\frac{e}{x^2} << \frac{d}{x}$. Therefore,

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx+e} = \lim_{x\to\infty}\frac{a}{d/x}$$

$$= \begin{cases} \frac{a}{0^+} \text{ if } d>0 & \begin{cases}+\infty \text{ if } a>0 \text{ and } d>0 \\ -\infty \text{ if } a<0 \text{ and } d>0\end{cases} \\ \frac{a}{0^-} \text{ if } d<0 & \begin{cases}-\infty \text{ if } a>0 \text{ and } d<0 \\ +\infty \text{ if } a<0 \text{ and } d<0\end{cases}\end{cases}$$

49. $f(x) = \lim_{n\to\infty}\tan^{-1}\left(4n^2\cdot 2\sin^2\frac{x}{2n}\right) = \lim_{n\to\infty}\tan^{-1}\left(8n^2\left(\frac{\sin\frac{x}{2n}}{\frac{x}{2n}}\right)^2\cdot\frac{x^2}{4n^2}\right) = \tan^{-1}(2x^2)$

$$g(x) = \lim_{n\to\infty}\frac{n^2}{2}\left(\frac{\ln\left(1+\cos^2\frac{2x}{n}-1\right)}{\cos^2\frac{2x}{n}-1}\right)\left(\cos\frac{2x}{n}-1\right) = x^2$$

50. $\lim_{x\to 0}\frac{\sin^2 x}{f(x)} = \frac{1}{3} \Rightarrow f(x) = x^2(ax+3);\quad a\neq 0$

51. $\lim_{x\to 0}\frac{(2e^{2\sin x}-e^{\sin x}-1)}{(x^2+2x)e^{\sin x}} = \lim_{x\to 0}\frac{(2e^{\sin x}+1)(e^{\sin x}-1)}{x(x+2)e^{\sin x}} = \frac{3}{2}$

52. $x^n + ax + b = (x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n)$

$$\lim_{x \to x_1} \frac{x^n + ax + b}{x - x_1} = (x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)$$

53. $$\lim_{x \to 0} \frac{\left(1 + \frac{1}{3}\sin^2 x + \ldots\right) - \left(1 - \frac{1}{4}(2\tan x) + \ldots\right)}{\sin x + \tan^2 x} = \frac{1}{2}$$

54. $$\lim_{x \to 0} \frac{f(x)}{x^2} = \lim_{x \to 0} \begin{vmatrix} \cos x & \frac{2\sin x}{x} & \tan x \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1$$

Exercise-2 : One or More than One Answer is/are Correct

1. $e^{\lim_{x \to 0} \frac{1}{3x^2}(p\tan qx^2 - 3\cos^2 x + 3)}$

$e^{\lim_{x \to 0} \frac{pq}{3} + \frac{3(1-\cos^2 x)}{3x^2}}$

$\Rightarrow \quad \frac{pq}{3} + 1 = \frac{5}{3}; \qquad pq = 2$

3. $a \geq e > 2$

(a) $L = a \lim_{x \to \infty} \left(1 + \left(\frac{2}{a}\right)^x + \left(\frac{e}{a}\right)^x\right)^{1/x}$

$\therefore \quad x \to a, \left(\frac{2}{a}\right) \to 0, \left(\frac{e}{a}\right) \to 0, \frac{1}{x} \to 0$

So, $L = a$

(b) If $a = 2e > 2$

$$L = \lim_{x \to \infty} (2^x + (2e)^x + e^x)^{1/x} = 2e \lim_{x \to \infty} \left[\left(\frac{1}{e}\right)^x + 1 + \left(\frac{1}{2}\right)^x\right]^{1/x} = 2e(1) = 2e$$

(c) If $0 < a \leq e$

$$L = e\left(\lim_{x \to \infty} \left(\left(\frac{2}{e}\right)^x + \left(\frac{a}{e}\right)^x + 1\right)^{1/x}\right) = e$$

(d) $a > \frac{e}{2} > 1$

$$L = \lim_{x\to\infty}\left[2^x + \left(\frac{2a}{2}\right)^x + e^x\right]^{1/x} = 2a \lim_{x\to\infty}\left(\left(\frac{1}{a}\right)^x + \left(\frac{1}{2}\right)^x + \left(\frac{e}{2a}\right)^x\right)^{1/x} = 0$$

5. $f(x) = \cos(\sin x)$

Range is $[\cos 1, 1]$.

8. $f(x) = x\left(\frac{3}{2} + \frac{3}{2}[\cos x]\right)$

9. If $x \neq \frac{1}{2^{2^n}}$ then $f(x) = 0$ but if $x = \frac{1}{2^{2^n}}$ then $\lim_{x\to 0} f(x) = \lim_{n\to\infty}(-1)^n$, hence does not exist.

Also, if $x = \frac{1}{2^{2^n}}$ then $2x \neq \frac{1}{2^{2^n}} \Rightarrow f(2x) = 0$

11. $$\lim_{x\to 0^+} \frac{\cos^{-1}(1-x)\sin^{-1}(1-x)}{\sqrt{2x}(1-x)} = \lim_{x\to 0^+} \frac{\left(\frac{\sin^{-1}\sqrt{2x-x^2}}{\sqrt{2x-x^2}}\right)\sqrt{2x-x^2}\cdot\sin^{-1}(1-x)}{\sqrt{2x}(1-x)} = \frac{\pi}{2}$$

$$\lim_{x\to 0^-} \frac{\cos^{-1}(-x)\sin^{-1}(-x)}{\sqrt{2(x+1)}(-x)} = \frac{\pi}{2\sqrt{2}}$$

12. $$\lim_{x\to 0} \frac{2\sin\left(\frac{\sin x - x}{2}\right)\cdot\cos\left(\frac{\sin x + x}{2}\right)}{ax^3 + bx^5 + c} = \frac{-1}{12}$$

$$\lim_{x\to 0} \frac{2\left(\frac{\sin\left(\frac{\sin x - x}{2}\right)}{\frac{\sin x - x}{2}}\right)\left(\frac{\sin x - x}{2}\right)\cdot\cos\left(\frac{\sin x + x}{2}\right)}{ax^3 + bx^5 + c} = \frac{-1}{12}$$

14. $\cos^2\left(n\pi + \frac{\pi}{3}\right)$

15. $\sin\alpha + \sin\beta = -\frac{\sin\beta}{\sin\alpha} \Rightarrow \sin\alpha = \sin\beta = -\frac{1}{2}$

16. $\lim_{x\to 2^+} [5 - 2x] = 0$

$\lim_{x\to 2^-} [|x-2| + a^2 - 6a + 9] = 0 \Rightarrow (a-3)^2 < 1$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $S_1 = 1, S_2 = 7, S_3 = 19$

$\Rightarrow \quad S_n = 1 + 3n(n-1)$

$\lim_{n\to\infty} \frac{S_n}{n^2} = 3$

2. $r_1 = 1, r_2 = \frac{1}{3}, r_3 = \frac{1}{5}$

or $\quad r_n = \frac{1}{2n-1}$

$\lim_{n\to\infty} n \times \frac{1}{2n-1} = \frac{1}{2}$

Paragraph for Question Nos. 3 to 4

3. $x > 0,\ x < \tan x$

$x < 0,\ x > \tan x \Rightarrow x - \tan x > 0$

$\therefore \quad [x - \tan x] = 0$

$\therefore \quad \lim_{x\to 0^-} f([x - \tan x]) = f(0) = 4$

4. $x > 0 \qquad x < \tan x$

$\frac{x}{\tan x} < 1$

$\lim_{x\to 0^+} \left\{\frac{x}{\tan x}\right\} = \frac{x}{\tan x} \to 1^-$

$\therefore \quad \lim_{x\to 0^+}\left(f\left\{\frac{x}{\tan x}\right\}\right) = \lim_{x\to 0^+} f\left(\frac{x}{\tan x}\right) = f(1^-) = 2 + 5 = 7$

Paragraph for Question Nos. 5 to 6

5. $f(x) = 1 - |x-2|$

$x \to 2^+, f(x) \to 1^-$ and $x \to 2^-, f(x) \to 1^-$

R.H.L. $= \lim_{x\to 2^+} (f(x))^{\frac{1}{\sin\left(\frac{\pi x}{2}\right)}} = e^{\lim_{x\to 2^+} \frac{f(x)-1}{\sin\left(\frac{\pi x}{2}\right)}}$

$$= e^{\lim_{x\to 2^+}\frac{1-(x-2)-1}{\sin\pi\left(1-\frac{x}{2}\right)}} = e^{\lim_{x\to 2^+}\frac{(x-2)}{\left(\frac{\sin\frac{\pi}{2}(2-x)}{\frac{\pi}{2}(2-x)}\right)}\times\frac{\pi}{2}(2-x)}$$

$$= e^{2/\pi}$$

$$\text{L.H.L.} = \lim_{x\to 2^-}(f(x))^{\frac{1}{\sin x\left(\frac{\pi x}{2}\right)}} = e^{\lim_{x\to 2^-}\frac{f(x)-1}{\sin\left(\frac{\pi x}{2}\right)}}$$

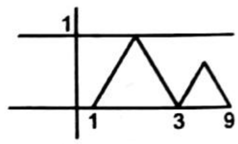
$$= e^{\lim_{x\to 2^-}\frac{1+(x-2)-1}{\sin\frac{\pi}{2}(2-x)}} = e^{\lim_{x\to 2^-}\frac{x-2}{\frac{\pi}{2}(2-x)}}$$

$$= e^{-2/\pi}$$

$\therefore$ Limit does not exist.

6. $[1, 3]$

as $f(3x) = \alpha f(x)$



$x \in [1, 3]$; $f(x) \in [0, 1]$

$3x \in [3, 9]$; $f(3x) = \alpha f(x) \in [0, \alpha]$

$9x \in [9, 27]$; $f(9x) = \alpha f(3x) \in [0, \alpha^2]$

area between $[1, 3]$ is $\Delta_1 = \frac{1}{2}\times 2\times 1 = 1$

area between $[3, 9]$ is $\Delta_2 = \frac{1}{2}\times 6\times\alpha = 3\alpha$

area between $[9, 27]$ is $\Delta_3 = \frac{1}{2}\times 18\times\alpha^2 = 9\alpha^2$

$\therefore$ $1, 3\alpha, 9\alpha^2, \ldots\ldots$ is converges when (g.p.) $|3\alpha| < 1$ $\alpha \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

Paragraph for Question Nos. 7 to 9

7. $$\lim_{x\to 0}\frac{[(1+bx)-(1+ax)\sqrt{1+x}]}{x^3} = \lim_{x\to 0}\frac{(1+bx)-(1+ax)\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{bx-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{16}-ax-\frac{ax^2}{2}+\frac{ax^3}{8}}{x^3}$$

$\Rightarrow$ coefficient of x and $x^2 = 0$ $\Rightarrow b - a = \frac{1}{2}$ and $\frac{a}{2} = \frac{1}{8}$

$\Rightarrow a = \frac{1}{4}, b = \frac{3}{4}$

8. $a + b = 1$

9. $l = -\frac{1}{32};\ b = \frac{3}{4}$

Paragraph for Question Nos. 10 to 11

Sol. $\sin x + \sin y = 1$

$$y' = \frac{-\cos x}{\sqrt{2\sin x - \sin^2 x}}$$

$$\Rightarrow \quad y'' = \frac{\sin^2 x - \sin x + 1}{(2\sin x - \sin^2 x)^{3/2}}$$

Exercise-5 : Subjective Type Problems

1. $$\lim_{x\to 0} \frac{\ln \tan\left(\frac{\pi}{4} - \beta x\right)}{\tan \alpha x} = -\lim_{x\to 0} \frac{\ln\left[\left(\frac{1-\tan\beta x}{1+\tan\beta x} - 1\right) + 1\right]}{\tan \alpha x}$$

$$= -1\left(-2\frac{\beta}{\alpha}\right) = 1$$

$$\Rightarrow \quad \frac{\alpha}{\beta} = 2$$

3. $a(x^3 - 1) + (x - 1) = 0$

$(x - 1)(ax^2 + ax + a + 1) = 0$

$\alpha, \beta \neq 1$ so, α, β are roots of $ax^2 + ax + a + 1 = 0$

$\alpha + \beta = -1,\ \alpha\beta = \frac{a+1}{a}$

$$\lim_{x\to\frac{1}{\alpha}} \frac{(1+a)x^3 - x^2 - a}{(e^{1-\alpha x} - 1)(x - 1)} = \lim_{x\to\frac{1}{\alpha}} \frac{(x^3 - x^2) + a(x^3 - 1)}{(e^{1-\alpha x} - 1)(x - 1)}$$

$$= \lim_{x\to\frac{1}{\alpha}} \frac{[x^2 + a(x^2 + x + 1)]}{(e^{1-\alpha x} - 1)} = \lim_{x\to\frac{1}{\alpha}} \frac{(1+a)x^2 + ax + a}{\left(\frac{e^{1-\alpha x} - 1}{1 - \alpha x}\right)(1 - \alpha x)}$$

$$= \lim_{x\to\frac{1}{\alpha}} a\frac{\left[\left(\frac{1+a}{a}\right)x^2+(1)x+1\right]}{(1-\alpha x)} = \lim_{x\to\frac{1}{\alpha}} a\frac{(\alpha\beta x^2-(\alpha+\beta)x+1)}{(1-\alpha x)}$$

$$= \lim_{x\to\frac{1}{\alpha}} a\frac{(1-(\alpha)x)(1-(\beta)x)}{(1-\alpha x)} = \frac{a(\alpha-\beta)}{\alpha}$$

4. $\lim\limits_{x\to 0}\dfrac{(4^x-1)(5^x-1)(7^x-1)}{x\sin^2 x} = 2\ln 2\ln 5\ln 7$

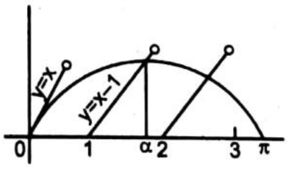
5. $\lim\limits_{x\to 0}\dfrac{ax\cos x+b\sin x}{x^2\sin x} = \dfrac{1}{3}$

$$\lim_{x\to 0}\frac{ax\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\dots\right)+b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\right)}{x^2\sin x} = \frac{1}{3}$$

$a+b=0$ and $-\dfrac{a}{2}-\dfrac{b}{6}=\dfrac{1}{3}$

$\Rightarrow b=1,\ a=-1$

7. $\lim\limits_{x\to\alpha^+}\left[\dfrac{\sin x}{x-1}\right]=0$



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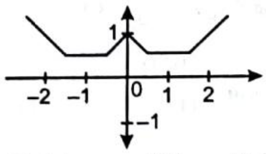
3 CONTINUITY, DIFFERENTIABILITY AND DIFFERENTIATION

Exercise-1 : Single Choice Problems

1. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + f(h) + 3hx(h+x) - f(x)}{h}$

$f'(x) = 3x^2 + f'(0) \Rightarrow f''(x) = 6x$

2. 

$f(x)$ is non-differentiable at five points.

3. $\frac{x}{5}$ is integer at 21 points in $[0, 100]$

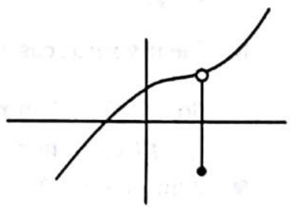
$\frac{x}{2}$ is integer at 51 points in $[0, 100]$

$\therefore$ But when x is a multiple of 10 then $f(x)$ is continuous.

So that respective points should be subtract from both *i.e.*, multiple of 10 are 11 points in $[0, 100]$.

$$21 + 51 - 11 - 11 = 72 - 22 = 50$$

4. $f(x)$ has isolated point of discontinuity but $|f(x)|$ is continuous at $x = a$



So, $\lim_{x \to a} f(x)$ and $f(a)$ has opposite sign, with same magnitude.

So, $\lim_{x \to a} f(x) = -f(a)$

$\lim_{x \to a} f(x) + f(a) = 0$

5. $\lim_{x \to 0} \frac{f(4x) - 3f(3x) + 3f(2x) - f(x)}{x^3} = 12$

$$\lim_{x\to 0}\frac{4f'(4x)-9f'(3x)+6f'(2x)-f'(x)}{3x^2}=12$$

$$\lim_{x\to 0}\frac{4^2f''(4x)-27f''(3x)+12f''(2x)-f''(x)}{6x}=12$$

$$\lim_{x\to 0}\frac{4^3f'''(4x)-81f'''(3x)+24f'''(2x)-f'''(x)}{6}=12$$

$\therefore$ $(4^3-81+24-1)f'''(0)=12\times 6$

$$6f'''(0)=12\times 6$$

$$f'''(0)=12$$

6. $y=\dfrac{1}{1+(\tan\theta)^{\sin\theta-\cos\theta}+(\tan\theta)^{\cot\theta-\cos\theta}}+\dfrac{1}{1+(\tan\theta)^{\cos\theta-\sin\theta}+(\tan\theta)^{\cot\theta-\sin\theta}}$

$$+\frac{1}{1+(\tan\theta)^{\cos\theta-\cot\theta}+(\tan\theta)^{\sin\theta-\cot\theta}}$$

$y=\dfrac{(\tan\theta)^{\cos\theta}}{(\tan\theta)^{\cos\theta}+(\tan\theta)^{\sin\theta}+(\tan\theta)^{\cot\theta}}+\dfrac{(\tan\theta)^{\sin\theta}}{(\tan\theta)^{\cos\theta}+(\tan\theta)^{\sin\theta}+(\tan\theta)^{\cot\theta}}$

$$+\frac{(\tan\theta)^{\cot\theta}}{(\tan\theta)^{\cos\theta}+(\tan\theta)^{\sin\theta}+(\tan\theta)^{\cot\theta}}$$

$y=1$

$$\left.\frac{dy}{dx}\right|_{0=\pi/3}=0$$

7. $f'(x)=\sin(x^2)$

$y=f(x^2+1)$

$$\frac{dy}{dx}=f'(x^2+1)2x$$

$$\left.\frac{dy}{dx}\right|_{x=1}=2\cdot f'(2)=2\sin 4$$

8. Clearly $\sin x,\cos x$ are negative at $x=\dfrac{7\pi}{6}$

So, $f(x)=-(\sin x+\cos x)$

$f'(x)=(\sin x-\cos x)$

9. $2\sin x\cos y=1$

$\cos x\cos y-\sin x\sin y\cdot y'=0 \Rightarrow y'_{(\pi/4,\,\pi/4)}=1$

$y'=\cot x\cot y$

$y''=-\cot x\,\text{cosec}^2 y\times y'-\cot y\,\text{cosec}^2 x$

$y''_{(\pi/4,\,\pi/4)}=-(1\times 2\times 1)-(1\times 2)=0$

10. $\frac{dx}{dt}=2t\,f'(t^2),\quad \frac{dy}{dt}=3t^2 f'(t^3)$

$$\frac{dy}{dx}=\frac{3}{2}\frac{tf'(t^3)}{f'(t^2)}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2}\left(\frac{f'(t^2)(f'(t^3)+3t^3f''(t^3))-2t^2f'(t^3)\cdot f''(t^2)}{(f'(t^2))^2}\right)\frac{dt}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1}=\frac{3}{2}\left(\frac{f'(1)(f'(1)+3f''(1))-2f'(1)\cdot f''(1)}{(f'(1))^2}\right)\frac{1}{2f'(1)}=\frac{3}{4}\left(\frac{f''(1)+f'(1)}{(f'(1))^2}\right)$$

11. L.H.L. $=a+1$

R.H.L. $=b+1$

$\therefore$ they are continuous L.H.L. = R.H.L.

12. $$y=\frac{\frac{1}{x}}{\frac{1}{x}-\alpha}+\frac{\frac{\beta}{x}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)}+\frac{\frac{\gamma}{x^2}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}$$

$$=\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)}+\frac{\frac{\gamma}{x^2}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}=\frac{\frac{1}{x^3}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}$$

$$\log y=-3\ln x-\ln\left(\frac{1}{x}-\alpha\right)-\ln\left(\frac{1}{x}-\beta\right)-\ln\left(\frac{1}{x}-\gamma\right)$$

$$\frac{1}{y}y'=\frac{-3}{x}+\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\alpha\right)}+\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\beta\right)}+\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\gamma\right)}$$

$$y'=\frac{y}{x}\left(-3+\frac{\frac{1}{x}}{\left(\frac{1}{x}-\alpha\right)}+\frac{\frac{1}{x}}{\left(\frac{1}{x}-\beta\right)}+\frac{\frac{1}{x}}{\left(\frac{1}{x}-\gamma\right)}\right)$$

$$y'=\frac{y}{x}\left(\frac{\alpha}{1/x-\alpha}+\frac{\beta}{1/x-\beta}+\frac{\gamma}{1/x-\gamma}\right)$$

13. $$f(x)=\sqrt{\frac{1+\sin^{-1}x}{1-\tan^{-1}x}}$$

$$\ln f(x)=\frac{1}{2}[\ln(1+\sin^{-1}x)-\ln(1-\tan^{-1}x)]$$

$$\frac{f'(x)}{f(x)}=\frac{1}{2}\left[\frac{1}{(1+\sin^{-1}x)\sqrt{1-x^2}}+\frac{1}{(1-\tan^{-1}x)(1+x^2)}\right]$$

$\therefore \quad f'(0)=1$

14. $\sin^2 x=-\sin^2 x \Rightarrow 2\sin^2 x=0 \Rightarrow x=n\pi$

15. $f(x)$ → $\tan x$ $\tan x<\cot x$

 → $\cot x$ $\tan x\geq\cot x$

Points of non-derivability $=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

16. $g(x)=|||x-1|-1|-1|$

$=x-3 \qquad x>3$

$=-(x-3) \qquad 2<x<3$

18. $$\frac{d^2x}{dy^2}=-\frac{1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx}=1+e^x,\ \frac{d^2y}{dx^2}=e^x$$

at $x=\ln 2,\ \frac{dy}{dx}=3,\ \frac{d^2y}{dx^2}=2$

$$\frac{d^2x}{dy^2}=\frac{-2}{27}$$

19. $g'(f(x))=\frac{1}{f'(x)}$

$f(x)=-4$ at $x=-2$

$\Rightarrow \quad g'(-4)=\frac{1}{f'(-2)}=\frac{1}{2}$

20. $f(x)=2-x \qquad x\geq 1$

$=x \qquad 0\leq x<1$

$=-x \qquad -1\leq x<0$

$=x+2 \qquad x<-1$

21. $f(x)=\cos x^2$

$f'(x)=-2x\sin x^2$

22. $f(g(x))=x \Rightarrow f'(g(x))g'(x)=1 \Rightarrow g'(x)=\frac{1}{f'(g(x))}=1+(g(x))^5$

$g''(x)=5(g(x))^4 g'(x)$

23. $f(x) = x^2 \qquad x \geq 1$

$= x \qquad 0 \leq x \leq 1$

$= 2x \qquad -1 \leq x \leq 0$

$= x - 1 \qquad x \leq -1$

Clearly it is non-differentiable at $x = 0, -1$ and 1.

24. $f(x) = \lim_{n\to\infty}\left(\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}\ldots\ldots\cos\frac{x}{2^n}\right) = \lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}$

25. $f\left(\frac{\pi^-}{4}\right) = f\left(\frac{\pi}{4}\right) = f\left(\frac{\pi^+}{4}\right)$

$$\lim_{x\to\frac{\pi}{4}}\left(\frac{1-\tan x}{4x-\pi}\right) = \lim_{x\to\frac{\pi}{4}}\frac{\tan\left(\frac{\pi}{4}-x\right)\cdot(1+\tan x)}{4\left(x-\frac{\pi}{4}\right)} = -\frac{1}{2}$$

26. $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}\sin\frac{1}{h}}{h}$

27. $\frac{dy}{dx} = 2y + 10$

$\int\frac{dy}{y+5} = 2\int dx$

$\ln(y+5) = 2x + c$

$y = 5(e^{2x} - 1) \qquad (\because c = \ln 5)$

$f(x) + 5\sec^2 x = 0 \Rightarrow e^{2x} + \tan^2 x = 0$

28. $f\left(\frac{\pi^-}{2}\right) = \lim_{x\to\frac{\pi^-}{2}}\frac{\sin\{\cos x\}}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi^-}{2}}\frac{\sin(\cos x)}{x-\frac{\pi}{2}} = -1$

$f\left(\frac{\pi^+}{2}\right) = \lim_{x\to\frac{\pi^+}{2}}\frac{\sin\{\cos x\}}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi^+}{2}}\frac{\sin(\cos x+1)}{x-\frac{\pi}{2}}$

29. Let $g(x) = f(e^x)$

$g'(x) = f'(e^x)\cdot e^x$

$g''(x) = f''(e^x)\cdot e^{2x} + f'(e^x)e^x$

30. $e^{f(x)} = \ln x \Rightarrow f(x) = \ln(\ln x) \Rightarrow g(x) = f^{-1}(x) = e^{e^x}$

$g'(x) = e^{e^x}\cdot e^x = e^{e^x+x}$

32. $\ln f(x) = 4\ln(x-1) + 3\ln(x-2) + 2\ln(x-3)$

$$\frac{f'(x)}{f(x)} = \frac{4}{x-1} + \frac{3}{x-2} + \frac{2}{x-3}$$

$$f'(x) = f(x)\left(\frac{4}{x-1} + \frac{3}{x-2} + \frac{2}{x-3}\right)$$

34. $f(2^+) = 0 \Rightarrow c = 0$

$$f(2^-) = \frac{b\sin\{-x\}}{\{-x\}} = f(2^+) = 0 \Rightarrow b = 0$$

35. $f(0) = \lim\limits_{x\to 0} \dfrac{e^{\tan x} - e^x + \ln(\sec x + \tan x) - x}{\tan x - x}$

$$= \lim_{x\to 0} e^x \frac{(e^{\tan x - x} - 1)}{\tan x - x} + \lim_{x\to 0} \frac{\ln(\sec x + \tan x) - x}{\tan x - x} = 1 + \lim_{x\to 0} \frac{\sec x - 1}{\sec^2 x - 1} = 1 + \frac{1}{2} = \frac{3}{2}$$

36. $f(0^-) = e^a$

$f(0) = b$

$c = 1$

$f(0^+) = \dfrac{2}{3} \Rightarrow b = e^a = \dfrac{2}{3}$

37. $\sqrt{x+y} + \sqrt{y-x} = 5$

$\sqrt{x+y} = 5 - \sqrt{y-x}$

Sq. both sides,

$\Rightarrow \quad x + y = 25 + y - x - 10\sqrt{y-x}$

$\Rightarrow \quad 25 - 2x = 10\sqrt{y-x}$

$\Rightarrow \quad -2 = \dfrac{10(y'-1)}{2\sqrt{y-x}}$

$\Rightarrow \quad -2\sqrt{y-x} = 5(y'-1)$

$\Rightarrow \quad -\left(5 - \dfrac{2x}{5}\right) = 5(y'-1)$

$\quad -5 + \dfrac{2x}{5} = 5(y'-1)$

$\Rightarrow \quad y'' = \dfrac{2}{25}$

38. $g(x) = f^{-1}(x)$

$\Rightarrow \quad f(g(x)) = x$

$\Rightarrow \quad f'(g(x))g'(x) = 1$

$\Rightarrow \quad g'(2) = \dfrac{1}{f'(g(2))}$

$f(1) = 2$

$\Rightarrow \quad g(2) = 1$

$\Rightarrow \quad g'(2) = \dfrac{1}{f'(1)}$

$f'(x) = 3x^2 + 4x^3 + \dfrac{1}{x}$

$f'(1) = 8$

$\Rightarrow \quad g'(2) = \dfrac{1}{8}$

39. $f(x) = |x| \qquad x \in (-\infty, -1)$

$= x^2 \qquad x \in [-1, 1)$

$= 2x - 1 \qquad x \in [1, \infty)$

Function is not differentiable at $x = -1$.

40. $g(x) = (f(x))^2 + (f'(x))^2 \Rightarrow g'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$

or $g'(x) = 2f(x)f'(x) - 2f(x)f'(x) = 0 \Rightarrow g(x) = c \Rightarrow g(8) = 8$

41. $l = \lim\limits_{x\to\infty}\left(f\left(\dfrac{a}{\sqrt{x}}\right)\right)^x = e^{\lim\limits_{x\to\infty}\left(\dfrac{f\left(\frac{a}{\sqrt{x}}\right)-1}{\frac{1}{x}}\right)}$

Using L Hospital's rule, we get

$l = e^{\frac{a^2}{2}f''(0)} = e^{-\frac{a^2}{2}}$

42. $\dfrac{d}{dx}f_n(x) = e^{f_{n-1}(x)}\dfrac{d}{dx}f_{n-1}(x) = f_n(x)\dfrac{d}{dx}f_{n-1}(x)$

$= f_n(x)f_{n-1}(x) \ldots\ldots f_2(x)f_1(x)$

43. $y = \tan^{-1}(x^{1/3}) - \tan^{-1}(a^{1/3})$

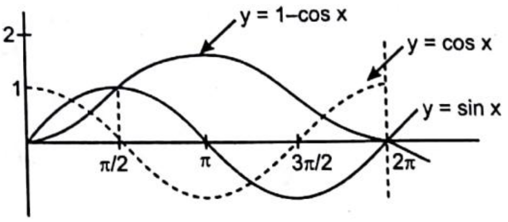
44. $f(x)$ is continuous at $x = 0$ then $\dfrac{4k-1}{3} = \dfrac{4k+1}{5}$

45. Put $x = \sin\theta$ then $y = \tan^{-1}\tan\dfrac{\theta}{2}$

46. $\lim\limits_{x\to 0}\dfrac{e^x\cos x - \ln(1+x) - 1}{x}$

$\lim\limits_{x\to 0}\dfrac{(e^x-1)}{x}\cos x - \dfrac{\ln(1+x)}{x} + \left(\dfrac{\cos x - 1}{x}\right) = 0$

47.



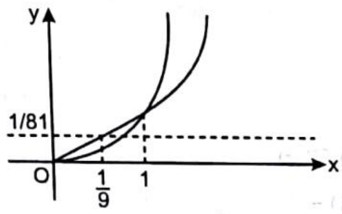
Clearly 3 sharp points.

48. $g(x) = f^{-1}(x)$ $\quad f(4) = 2 \Rightarrow g(2) = 4$

$G(x) = \dfrac{1}{g(x)}$ $\quad f'(4) = \dfrac{1}{16} \Rightarrow g'(2) = 16$

$$G'(x) = \frac{-1}{(g(x))^2} \cdot g'(x) \Rightarrow G'(2) = \frac{-1}{(g(x))^2} \cdot g'(2) = \frac{-1}{16} \cdot 16 = -1$$

49.



$$f(x) = \text{maximum}\left(x^4, x^2, \frac{1}{81}\right) = \frac{1}{81} \qquad x \le \frac{1}{9}$$

$$= x^2 \qquad \frac{1}{9} < x < 1$$

$$= x^4 \qquad x \ge 1$$

$f(x)$ is non-differentiable at $x = \dfrac{1}{9}, 1$

50. $\displaystyle \lim_{h \to 0} \frac{\ln(f(2+h^2)) - \ln(f(2-h^2))}{h^2}$

Apply L' Hospital rule,

$$\lim_{h \to 0} \frac{\dfrac{2hf'(2+h^2)}{f(2+h^2)} + \dfrac{2hf'(2-h^2)}{f(2-h^2)}}{2h} = 4$$

51. $f(x) = (x^2 - 3x + 2)|(x-1)(x-2)(x-3)| + \left|\sin\left(x + \dfrac{\pi}{4}\right)\right|$

Not differentiable at $x = 3, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

52. $h(x) = f(2x\,g(x) + \cos \pi x - 3)$

$h'(x) = f'(2x\,g(x) + \cos \pi x - 3)[2g(x) + 2xg'(x) - \pi \sin \pi x]$

$h'(1) = f'(2g(1) - 4)[2g(1) + 2g'(1)] = 32$

53. $f(x)=\dfrac{(x+1)^7\sqrt{1+x^2}}{(x^2-x+1)^6}$ $\quad (f(0)=1)$

$$\ln f(x)=7\ln(1+x)+\frac{1}{2}\ln(1+x^2)-6\ln(x^2-x+1)$$

$$\frac{f'(x)}{f(x)}=\frac{7}{1+x}+\frac{x}{1+x^2}-\frac{6(2x-1)}{x^2-x+1}$$

$$f'(0)=13$$

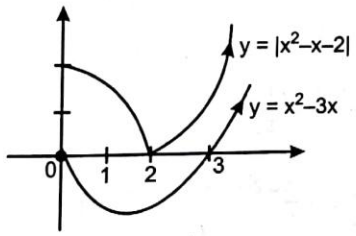
54. $f(x)\begin{cases}-\sin 2x; & x>1\\ \ln(1+x); & x<1\\ \dfrac{\ln 2-\sin 2}{2}; & x=1\end{cases}$; $\quad f(1^+)\neq f(1^-)\neq f(1)$

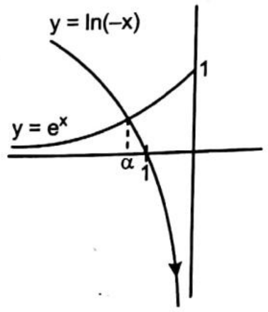
55. $f(f(x))\begin{bmatrix}f(x); & \text{if } f(x) \text{ is rational}\\ 1-f(x); & \text{if } f(x) \text{ is irrational}\end{bmatrix}$

$f(f(x))\begin{bmatrix}x; & \text{if } x \text{ is rational}\\ 1-(1-x); & \text{if } x \text{ is irrational}\end{bmatrix}$

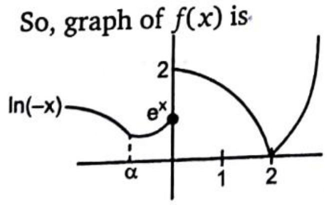
$f(f(x))\begin{bmatrix}x; & \text{if } x \text{ is rational}\\ x; & \text{if } x \text{ is irrational}\end{bmatrix}$

56.





So, graph of $f(x)$ is



Clearly, 3 non-differentiability points.

57. $g(f(x)) = x$

58. $\lim\limits_{x\to 0^-} \dfrac{\ln(2-\cos 2x)}{\ln^2(1+\sin 3x)} = K = \lim\limits_{x\to 0^+} \dfrac{e^{\sin 2x}-1}{\ln(1+\tan 9x)}$

$\lim\limits_{x\to 0^-} \dfrac{1-\cos 2x}{\sin^2 3x} = K = \lim\limits_{x\to 0^+} \dfrac{\sin 2x}{\tan 9x}$

59. $\dfrac{dx}{dt} = -\dfrac{3}{t^4} - \dfrac{2}{t^3} = \dfrac{-3-2t}{t^4}$

$\dfrac{dy}{dt} = \dfrac{-3}{t^3} - \dfrac{2}{t^2} = \dfrac{-3-2t}{t^3}$

$\dfrac{dy}{dx} = t$

$\dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^3 = t - \left(\dfrac{1+t}{t^3}\right)\cdot t^3 = -1$

60. $-\dfrac{2}{y^3} y' = 2\sqrt{2}(-2\sin 2x)$

$\dfrac{(y')^2}{y^6} = 8 - (2\sqrt{2}\cos x)^2 = 8 - \left(\dfrac{1}{y^2} - 1\right)^2$

$\dfrac{(y')^2}{y^6} = \dfrac{8y^2 - (1-y^2)^2}{y^4}$

$(y')^2 = 8y^4 - y^2(1-y^2)^2$ then diff.

61. $f(x) = x$ satisfy the equation.

$\therefore \quad f(5) = 5$

62. $f(x)\begin{cases} x & x \le 0 \\ x^2 & 0 < x < 1 \\ 2x-1 & x \ge 1 \end{cases}$

$f'(x)\begin{cases} 1 & x \le 0 \\ 2x & 0 < x < 1 \\ 2 & x \ge 1 \end{cases}$

$f(x)$ is not derivable at $x = 0$.

63. $y = (x + \sqrt{1+x^2})^n$

$\dfrac{dy}{dx} = \dfrac{ny}{\sqrt{1+x^2}}$

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}\,y'-\frac{yx}{\sqrt{1+x^2}}}{1+x^2}\right] \Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

64. $g'(x)=f'(x-\sqrt{1-x^2})\cdot\left(1+\frac{x}{\sqrt{1-x^2}}\right)=\left(1-\left(x-\sqrt{1-x^2}\right)^2\right)\cdot\left(\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$

$$=2x\left(x+\sqrt{1-x^2}\right)$$

66. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}f(x)\left(\frac{f(h)-1}{h}\right)=f(x)\cdot f'(0)=3f(x)$ $\quad(\because\ f'(0)=3)$

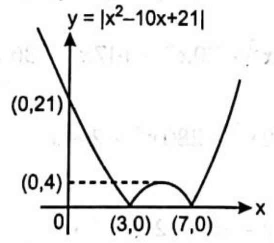
67. $f(x)=\lim_{n\to\infty}\frac{\log_e(2+x)-x^{2n}\sin x}{1+x^{2n}}$

$$f(x)=\begin{cases}\ln(2+x) & |x|<1\\ -\sin x & |x|>1\\ \frac{\ln 3-\sin 1}{2} & x=1\\ \frac{\sin 1}{2} & x=-1\end{cases}$$

68. $\lim_{x\to 0}\frac{x-e^x+1-\{1-\cos 2x\}}{x^2}\Rightarrow\lim_{x\to 0}\frac{x-e^x+1-1+\cos 2x}{x^2}$

$$\Rightarrow\lim_{x\to 0}\frac{x-e^x+\cos 2x}{x^2}=\lim_{x\to 0}\frac{1+x-e^x}{x^2}+\frac{(\cos 2x-1)}{x^2}=-\frac{5}{2}$$

69.



71. $xy=\text{const.}$

$y+xy'=0\Rightarrow y'=-\frac{y}{x}$

72. $f(x)=-1+|x-2|$ is a continuous function.

$g(x) = 1 - |x|$ is a continuous function.

$\Rightarrow$ $f(g(x))$ is a continuous function.

73. $f'(K^+) = \lim_{h\to 0} \dfrac{f(k+h) - f(k)}{h}$

$$= \lim_{h\to 0} \frac{K\tan(\pi k + \pi h) - k\tan k\pi}{h}$$

$$= \lim_{h\to 0} k\left(\frac{\tan \pi h}{h}\right) = k\pi$$

74.
$$\lim_{x\to 0} \frac{ae^{\sin x} + be^{-\sin x} - c}{x^2} = 2$$

$$a + b - c = 0 \qquad \ldots(1)$$

Applying L Hospital Rule,

$$\lim_{x\to 0} \frac{ae^{\sin x}\cdot\cos x - be^{-\sin x}\cdot\cos x}{2x} = 2 \Rightarrow a = b$$

75. $\tan x = \sec\alpha\cdot\tan y$

$\sec^2 x = \sec\alpha\cdot\sec^2 y\cdot y'$

$y' = 1 \text{ at } \left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

$2\sec^2 x\tan x = \sec\alpha\,(\sec^2 y\cdot y'' + 2\sec^2 y\cdot\tan y\cdot(y')^2) \Rightarrow y'' = 0$

76. We gave,
$$y = (x^2-9)(x^2-4)(x^2-1)\,x$$
$$= \{x^6 - 14x^4 + x^2(49) - 36\}\,x$$
$$= x^7 - 14x^5 + 49x^3 - 36x$$

Therefore,
$$\frac{dy}{dx} = 7x^5 - 70x^4 + 147x^2 - 36$$

Thus,
$$\frac{d^2y}{dx^2} = 42x^5 - 280x^3 + 294x$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 42 - 280 + 294 = 56$$

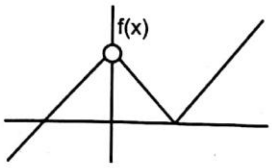
77. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{f(x)\,f(h) - f(x)}{h} = \lim_{h\to 0} f(x)\left(\dfrac{f(h)-1}{h}\right)$

$\Rightarrow$ $f'(x) = f'(0),\ f(x) \Rightarrow f(x) = e^{kx}$ (where $k = f'(0)$)

78. $f(g(x)) = x$; $\quad f'(g(x))g'(x) = 1 \Rightarrow g'(6) = \dfrac{1}{f'(g(6))} = \dfrac{1}{f'(0)}$

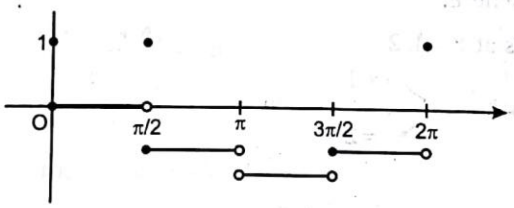
79. $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{\frac{d}{dx}\left(\frac{dy}{dz}\right)}{\frac{dz}{dx}} = \frac{g'f'' - f'g''}{(g')^3}$$

80. 

$$g(f(x)) = \begin{cases} f(x) + 1 = x + 2, x \in (-\infty, -1) \; x = -1, 1 \text{non differentiable} \\ (f(x) - 1)^2 = (x + 1 - 1)^2 = x^2, x \in (-1, 0) \\ (|x - 1| - 1)^2, x \geq 0 \end{cases}$$

81. $f(x) = [\sin x] + [\cos x]$



82. $g(x) = \begin{cases} \cos x & , x \in [0, \pi] \\ \sin x - 1 & , x > \pi \end{cases}$

$g(\pi^-) = g(\pi) = g(\pi^+) = -1$

but not differentiable at $x = \pi$.

83. $\displaystyle\sum_{r=0}^{\infty} \frac{f^r(0)}{r!} = \frac{f(0)}{0!} + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \ldots$

$$= 4^n + \frac{n \cdot 4^{n-1}}{1!} + \frac{n(n-1) \cdot 4^{n-2}}{2!} + \ldots$$

$$= {}^nC_0 4^n + {}^nC_1 4^{n-1} + {}^nC_2 4^{n-2} + \ldots$$

$$= (4 + 1)^n = 5^n$$

84. $f(x) = \dfrac{x}{1-x} \qquad x \le -1$

$= \dfrac{x}{1+x} \qquad -1 < x < 0$

$= \dfrac{x}{1-x} \qquad 0 \le x < 1$

$= \dfrac{x}{1+x} \qquad x \ge 1$

Function is discontinuous at $x = -1, 1$

$f(x)$ is not differentiable at $x = -1, 1$

85. $f(g(x)) = x$

$f'(g(x))g'(x) = 1 \Rightarrow g'\left(\dfrac{-7}{6}\right) = \dfrac{1}{f'\left(g\left(\dfrac{-7}{6}\right)\right)} = \dfrac{1}{f'(1)}$

86. $f(x) = 0 \qquad x \ge 0$

$= 4x^2(1-2x)^2 \qquad x < 0$

Differentiable everywhere.

88. $f(x)$ is discontinuous at $x = 1, 2$

$\Rightarrow \quad g(x) = x^2 - ax + b = 0 \begin{cases} x=1 \\ x=2 \end{cases}$

89. $f^{-1}(f(x)) = x$

$(f^{-1}(f(x)))'f'(x) = 1$

$(f^{-1}(f(9)))'f'(g) = 1$

$(f^{-1}(3))' = \dfrac{1}{f'(9)} = \dfrac{1}{5}$

90. $f(0^+) = \lim\limits_{h \to 0} h^n \sin\dfrac{1}{h} = 0 \Rightarrow n > 0$

$f(0^-) = \lim\limits_{h \to 0} (-h)^n \sin\left(-\dfrac{1}{h}\right) = 0 \Rightarrow n > 0$

$f'(x) = n \cdot x^{n-1} \cdot \sin\dfrac{1}{x} - x^{n-2} \cdot \cos\dfrac{1}{x} = \text{finite} \Rightarrow n = 2$

Exercise-2 : One or More than One Answer is/are Correct

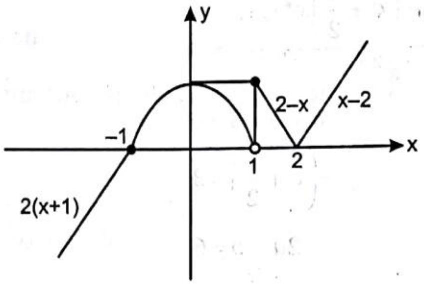
1. $f(x)$ has exactly one point of discontinuity so that sgn $(x^2 - \lambda x + 1)$ is equal to zero for some values of λ.

$\therefore \quad D = 0$

$\Rightarrow \quad \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda = \pm 2$

2. Answer from the graph.



3. (a) L.H.L. $= \lim_{x \to 0^-} \dfrac{x(3e^{1/x} + 4)}{2 - e^{1/x}} = 0\left(\dfrac{4}{2}\right) = 0$

R.H.L. $= \lim_{x \to 0^+} \dfrac{x(3e^{1/x} + 4)}{2 - e^{1/x}} = \lim_{x \to 0^+} x\left(\dfrac{3 + 4e^{-1/x}}{2e^{-1/x} - 1}\right) = 0\left(\dfrac{3}{-1}\right) = 0$

$f(0) = 0$

$\therefore \quad f(x)$ is continuous at $x = 0$.

(b) $f'(0^+) = \lim_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \dfrac{x\left(\dfrac{3e^{1/x} + 4}{2 - e^{1/x}}\right)}{x}$

$= \lim_{x \to 0^+} \dfrac{3 + 4e^{-1/x}}{2e^{-1/x} - 1} = -3$

$f'(0^-) = \lim_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \dfrac{3e^{1/x} + 4}{2 - e^{1/x}} = \dfrac{4}{2} = 2$

$f'(0^+) \neq f'(0^-)$

(c) $f'(0^+) = -3$

(d) $f'(0^-) = 2$ exist

4. Given $|f(x)| \leq \sin^2 x$

Clearly $|f(0)| \leq 0 \Rightarrow f(0) = 0$

$$\lim_{x\to 0}|f(x)| = \left|\lim_{x\to 0} f(x)\right| = 0$$

$$|f'(0)| = \left|\lim_{x\to 0}\frac{f(x)-f(0)}{x}\right| = \left|\lim_{x\to 0}\frac{f(x)}{x}\right| \le 0$$

5. $$f(0^-) = \lim_{x\to 0^-}\frac{a\left[1-x\left(x-\frac{x^3}{3!}\ldots\right)\right]+b\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)+5}{x^2} = f(0)$$

$$= \lim_{x\to 0^-}\frac{(a+b+5)-\left(a+\frac{b}{2}\right)x^2+\ldots}{x^2} = 3$$

$$a+b+5=0 \qquad \ldots(1)$$

$$-\left(a+\frac{b}{2}\right)=3$$

$$2a+b=6 \qquad \ldots(2)$$

On solving (1) and (2),

$$a+b+5=0$$

$$2a+b+6=0$$

$$\underline{-\quad -\quad -}$$

$$-a-1=0$$

$$a=-1$$

$$b=-4$$

$\therefore$ $$a+b=-5$$

$f'(0^+)$ is exist when $c=0$

$$\lim_{x\to 0}(1+dx)^{1/x}=3$$

$$e^{\lim_{x\to 0}\frac{1}{x}(dx)}=3$$

$\Rightarrow$ $$e^d=3$$

$$d=\ln 3$$

7. (a) $f(x)=\sqrt[3]{x^2|x|}-1-|x|$

But $x^2|x|=|x|^3$

So, $f(x)=|x|-1-|x|=-1$ is every where differentiable.

So, no where non-differentiable.

(b) $\lim_{x\to\infty} [x(\tan^{-1}(x+1) - x\tan^{-1}(x+1))] + [5\tan^{-1}(x+1) - \tan^{-1}(x+1)]$

$$= \lim_{x\to\infty} 4\tan^{-1}(x+1) = 4\left(\frac{\pi}{2}\right) = 2\pi$$

(c) $f(-x) = \sin\left(\ln\left(-x+\sqrt{x^2+1}\right)\right) = \sin\left(\ln\frac{1}{x+\sqrt{x^2+1}}\right)$

$= \sin\left(-\ln\left(x+\sqrt{x^2+1}\right)\right) = -\sin\left(\ln\left(x+\sqrt{x^2+1}\right)\right)$

$= -f(x)$

So, $f(x)$ is an odd function.

(d) $f(x) = \dfrac{4-x^2}{4x-x^3}$ is discontinuous at where denominator is zero, $4x - x^3 = 0$

$\Rightarrow \quad x = 0, x = \pm 2$

$\therefore \quad$ a, b, c only correct.

8. $g'(x) = ae^{ax} + f'(x) \Rightarrow g'(0) = a - 5$

$g''(x) = a^2e^{ax} + f''(x)$

$g''(0) = a^2 + 3$

$\Rightarrow \quad a^2 + a - 2 = 0;\quad a = -2, 1$

10. $f(0^+) = f(0) = f(0^-) = 0$

11. $\int f'(x)\,dx = \int f'(-x)\,dx$

$\Rightarrow \quad f(x) + f(-x) = c$

12. $|f(x)| \le x^{4n}$

$\Rightarrow \quad f(0) = 0$

$\lim_{h\to 0}(-h^{4n}) \le \lim_{h\to 0} f(0+h) \le \lim_{h\to 0}(h)^{4n} \quad \Rightarrow f(0+h) = 0$

$\lim_{h\to 0}(-(-h)^{4n}) \le \lim_{h\to 0} f(0-h) \le \lim_{h\to 0}(-h)^{4n} \Rightarrow f(0-h) = 0$

$\Rightarrow \quad f(x)$ is continuous at $x = 0$.

$f'(0^+) = \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = 0 \quad \left[\lim_{h\to 0}\dfrac{-h^{4n}}{h} \le \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h} \le \lim_{h\to 0}\dfrac{h^{4n}}{h}\right]$

$\Rightarrow \quad f(x)$ is differentiable at $x = 0$.

13. $g(x) = 0 \qquad x \in I$

$= x^2 \qquad x \notin I$

$gof(x) = 0$ for $x \in R$

14. If $f(x)$ is continuous at $x = 2$ then $3p + 10q = 4$

$f(x)$ is differentiable at $x = 2$ then $2p + 11q = 4$

16. $f(x) = x^2 \quad -2 \le x \le 0$

$= x \quad 0 < x < 1$

$= x^3 \quad 1 \le x \le 2$

17. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f(x) \lim_{h \to 0} \frac{\frac{f(x+h)}{f(x)} - 1}{h}$

$$= f(x) \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - 1}{h}$$

So, $f'(x) = \frac{f(x)}{x} \cdot f'(1)$

$\ln(f(x)) = 3 \ln x + \ln c$

$f(x) = cx^3$

$f(1) = 1$ so $c = 1$

$f(x) = x^3$

So, we can check options.

18. $f(x) = (x-1)(x-2)(x+1)(x+2) = (x^2 - 1)(x^2 - 4)$

$f'(x) = (x^2 - 1)2x + (x^2 - 4)(2x) = 2x(2x^2 - 5) = 0$

$x = 0, \pm\sqrt{\frac{5}{2}}$

19. If $f(x)$ is continuous at $x = 2$ then $3p + 10q = 4$

$f(x)$ is differentiable at $x = 2$ then $2p + 11q = 4$

20. $y = e^{x \sin x^3} + e^{x \ln(\tan x)}$

$\frac{dy}{dx} = e^{x \sin(x^3)}[x \cos(x^3) 3x^2 + \sin(x^3)] + e^{x \ln(\tan x)}\left(\ln(\tan x) + x \frac{1}{\tan x} \sec^2 x\right)$

$y' = e^{x \sin(x^3)}[3x^3 \cos(x^3) + \sin(x^3)] + (\tan x)^x (\ln(\tan x) + 2x \operatorname{cosec} 2x)$

21. $f(x) = 1 - (1-x) + (1-x)x^2 + (1-x)(1-x^2)x^3 + \ldots + (1-x)(1-x^2)\ldots(1-x^{n-1})x^n$

$$= 1 - (1-x)(1-x^2)(1-x^3)\ldots(1-x^n) = 1 - \prod_{r=1}^{n}(1-x^r)$$

$(f(x) - 1) = -\prod_{r=1}^{n}(1-x^r)$

22. $\therefore$ f and g must be continuous.

$1+a=2+b$

$\Rightarrow a=1+b$

$3+b=1 \Rightarrow b=-2$

$a=-1$

23. $$f(x)=\begin{cases} ax^3+b\,; & 0\le x\le 1 \\ 2\cos \pi x+\tan^{-1}x\,; & 1<x\le 2 \end{cases}$$

is must be continuous and differentiable at $x=1$.

$\therefore$ $a+b=-2+\frac{\pi}{4}$ ...(1) (continuity)

$3a=0+\frac{1}{2}$ ...(2) (By differentiable)

We get, a and b

24. $f(f(x))=2+x$ $\quad 0\le x\le 1$

$=2-x$ $\quad 1<x\le 2$

$=4-x$ $\quad 2<x\le 3$

25. $\ln(f(x))=\ln(x+1)+\ln(x+2)+......+\ln(x+100)$

$$\frac{f'(x)}{f(x)}=\frac{1}{x+1}+\frac{1}{x+2}+......+\frac{1}{x+100}$$

$$\frac{f(x)f''(x)-(f'(x))^2}{(f(x))^2}=-\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+......+\frac{1}{(x+100)^2}\right)$$

if $g(x)=f(x)f''(x)-(f'(x))^2=0$

$$\Rightarrow \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+......+\frac{1}{(x+100)^2}=0$$

$\Rightarrow g(x)=0$ has no solution.

26. $h(x)=-1$ $\quad x<1$

$=|x-2|+a+2-|x|$ $\quad 1\le x<2$

$=|x-2|+a+1-b$ $\quad x\ge 2$

if $h(x)$ is continuous at $x=1$, then $a=-3$

if $h(x)$ is continuous at $x=2$, then $b=1$

27. $\lim\limits_{x\to 0^-} f(x)=1=\lim\limits_{x\to 0^+} f(x)$

Clearly, $C=1$ and use L' Hospital's rule.

28. Differentiable w.r.t. 'x'

$$2f(x)f'(x)+2y=2f(x+y)f'(x+y)$$

put $x = 0$

$$k + y = f'(y)f(y)$$

integrate on both sides,

$$ky + \frac{y^2}{2} = \frac{f^2(y)}{2} + c \quad \ldots(1)$$

put $x = y = 0$ in given equation, we get

$$f^2(0) = 2$$

$f(0) = \sqrt{2}$ as $(f(x) > 0)$

put $y = 0$ in (1)

$$1 + c = 0 \Rightarrow c = -1$$

also put $y = \sqrt{2}$

$$k\sqrt{2} + 1 = \frac{4}{2} - 1$$

$$k\sqrt{2} = 0$$

$$k = 0$$

$$\therefore \quad \frac{y^2}{2} = \frac{f^2(y)}{2} - 1$$

$$f^2(y) = y^2 + 2$$

$$f(y) = \sqrt{y^2 + 2}$$

$$f(x) = \sqrt{x^2 + 2}$$

Hence, we can answer.

30. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{f(h)}{h} + x^2 = x^2 + f'(0)\,;\ f'(x) = x^2 - 1$

32. $f(1^-) = f(1^+) = f(1) = \dfrac{1}{2}$

$f'(x) = x \qquad 0 \le x < 1$

$\quad = 4x - 3 \qquad 1 \le x \le 2$

$f''(x) = 1 \qquad 0 \le x < 1$

$\quad = 4 \qquad 1 \le x \le 2$

34. $gof(x) = 0$

$fog(x) = 0 \qquad x \in I$

$\quad = [x^2] \qquad x \notin I$

35. $f(g(x)) = x$

$f'(g(x))g'(x) = 1$

$g'(x) = \dfrac{1}{f'(g(x))}$

$g'(e) = \dfrac{1}{f'(g(e))} = \dfrac{1}{f'(1)} = \dfrac{1}{e+1}$

$g''(x) = \dfrac{-1}{(f'(g(x)))^2} f''(g(x)) \cdot g'(x)$

$g''(e) = \dfrac{-1}{(f'(1))^2} f''(1) \cdot g'(e)$

36. $f(2^+) = \lim\limits_{x \to 2^+} [x-1] = 1$

$f(2^-) = \lim\limits_{x \to 2^-} \dfrac{3x - x^2}{2} = 1$

$f(3^-) = \lim\limits_{x \to 3^-} [x-1] = 1$

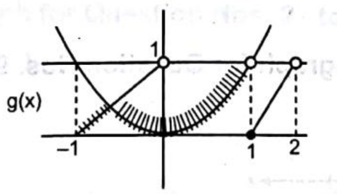
$f(3^+) = \lim\limits_{x \to 3^+} (x^2 - 8x + 17) = 2$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

2. $f(x) = \lim\limits_{n \to \infty} n^2 \dfrac{\tan(\ln(\sec(x/n)))}{\ln(\sec(x/n))} \times \dfrac{(\ln(\sec(x/n) - 1) + 1)}{\sec(x/n) - 1} \times \dfrac{\sec(x/n) - 1}{(x/n)^2} \times \left(\dfrac{x}{n}\right)^2$

$f(x) = \dfrac{x^2}{2}$



Paragraph for Question Nos. 3 to 4

4. $f'(x) = 2x + g'(1)$

$f''(x) = 2$

$f'(1) = 2 - 3 = -1$

$g'(1) = 2f(1) + 2 + f'(1)$

$\Rightarrow \quad f(1) = -2$

$f(x) = x^2 - 3x$

$g''(2) = 2(-2) + 2(2) = 0$

$-2 = 1 + g'(1)$

$g(x) = -2x^2 + x(2x-3) + 2$

$g'(1) = -3$

$= -3x + 2$

$f(1) + g(-1) = -2 + (3+2) = 3$

Paragraph for Question Nos. 5 to 6

5. Clearly, 3 is non-repeated root where as 1 is repeats and also $(x-2)^{1/3}$ is not diff. at $x = 2$.

$\therefore$ at 3, 2 is non-diff. and sum is 5.

6. $h(x)$ is continuous.

$x - 1 = x^2 - x - 2$

$x^2 - 2x - 1 = 0$

$(x-1)^2 = 2$

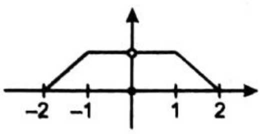
$x = 1 \pm \sqrt{2}$

$\tan\frac{3\pi}{8} = 1 + \sqrt{2}, \tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$

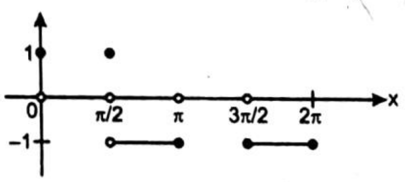
$\tan\frac{7\pi}{8} = 1 - \sqrt{2}$

$\sqrt{2} - 1$ is not differentiable.

Paragraph for Question Nos. 7 to 8

Sol. 

Paragraph for Question Nos. 9 to 10

10. 

Paragraph for Question Nos. 11 to 13

11. $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ x & 1 \le x < 2 \\ 2(x-1) & 2 \le x < 3 \\ 3(x-1) & x = 3 \end{cases}$

No. of values where $f(x)$ is discontinuous $= 2$

12. $f(x)$ is non-differentiable at $x = 1, 2, 3$.

13. No. of integers in the range of $f(x) = 5$

Paragraph for Question Nos. 14 to 16

Sol. $f'(x) = \lim_{h \to 0} \dfrac{f(x)f(h) - f(x)}{h} = f'(0) f(x)$

$\Rightarrow \quad f(x) = e^{2x} \qquad (f'(0) = 2)$

$g(x) = x^2$

Paragraph for Question Nos. 17 to 18

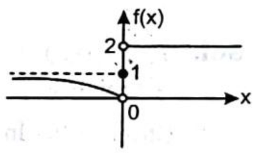
Sol. $g'(x) = \lambda \sec^2 x + (1 - \lambda)\cos x - 1 = \dfrac{(1 - \cos x)(\lambda - f(x))}{f(x)}$

Paragraph for Question Nos. 19 to 21

Sol. $f(x) = \dfrac{x^2}{x^2 + 1} \qquad x < 0$

$= 1 \qquad x = 0$

$= 2 \qquad x > 0$



Paragraph for Question Nos. 22 to 24

Sol. $f(x) = g'(1)\sin x + (g''(2) - 1)x$

$\Rightarrow \quad f'(x) = g'(1)\cos x + g''(2) - 1 \Rightarrow f'\left(\dfrac{\pi}{2}\right) = g''(2) - 1$

$f''(x) = -g'(1)\sin x \qquad \Rightarrow f''\left(\dfrac{\pi}{2}\right) = -g'(1)$

$g(x) = x^2 - f'\left(\dfrac{\pi}{2}\right) \cdot x + f''\left(-\dfrac{\pi}{2}\right) \qquad \Rightarrow g'(x) = 2x - f'\left(\dfrac{\pi}{2}\right)$

$g''(x) = 2 \qquad \Rightarrow g''(2) = 2$

$f(x) = \sin x + x$ and $g(x) = x^2 - x + 1$

Paragraph for Question Nos. 25 to 26

Sol. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x)+f(h)}{1+f(x)f(h)} - f(x)}{h}$

$f'(x) = \lim_{h\to 0} f(h)\frac{(1-(f(x))^2)}{h}$

$f'(x) = \lim_{h\to 0} f'(0)(1-f(x)^2) \qquad (f(0)=0)$

$\Rightarrow \; f(x) = \frac{e^{2x}-1}{e^{2x}+1}$

$f'(x) \ge 0 \; \forall \; x \in R$

$\lim_{x\to 0}(f(x))^x = e^{\lim_{x\to 0}\frac{-2x}{(e^{2x}+1)}} = 1$

Paragraph for Question Nos. 27 to 28

Sol. $f(x) = 3(x+6)(x+1)(x-2)(x-3) + x^2 + 1$ then k why!

27. $\lim_{x\to -6} \frac{3(x+1)(x-2)(x-3)(x+6)}{x+6} = -\frac{6!}{2}$

28. $g(x) = \frac{1}{-3(x+6)(x+1)(x-2)(x-3)}$

Paragraph for Question Nos. 29 to 30

Sol. $f(x) = g(x)$

$x^{\ln x} = e^2 x$

$(\ln x)^2 = 2 + \ln x$

$x = \alpha = \frac{1}{e}, \; \beta = e^2$

29. $\lim_{x\to e^2} \frac{f(x)-c\beta}{g(x)-\beta^2} = \frac{f'(x)}{g'(x)} = 4$

$c = e^2$

30. $h'(\alpha) = \frac{g(\alpha)\,f'(\alpha) - g'(\alpha)f(\alpha)}{g^2(\alpha)} = \frac{e(-2e^2)-e^2-e}{(e)^2} = -3e$

Exercise-4 : Matching Type Problems

1. (A) Let $I = \int_0^{\pi} \frac{\log(\sin x)}{\cos^2 x} dx$

$$= 2\int_0^{\pi/2} \frac{\log(\sin x)}{\cos^2 x} dx = 2\left[\int_0^{\pi/2} \log(\sin x)\sec^2 x\, dx\right] = 2\left[\log(\sin x)\tan x\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos x}{\sin x}\tan x\, dx\right]$$

$$= 2(0-0) - 2\int_0^{\pi/2} dx = 0 - 2\left(\frac{\pi}{2}\right) = -\pi = -k$$

$\Rightarrow \quad k = \pi$

$\therefore \quad \frac{3k}{\pi} = 3 > 0, 1, 2$

(B) $e^{x+y} + e^{y-x} = 1$

$e^x + e^{-x} = e^{-y}$

$e^x - e^{-x} = e^{-y}(-y')$

$e^x + e^{-x} = e^{-y}(-y'') + e^{-y}(y')^2$

$e^{-y} = e^{-y}(-y'') + e^{-y}(y')^2 \Rightarrow y' - (y')^2 + 1 = 0$

$\therefore \quad k = 1$

(C) Let $f^{-1} = g$

$g\{f(x)\} = x \Rightarrow (g'f(x))f'(x) = 1$

$g'(2\ln 2)\, f'(2) = 1$

$g'2(\ln 2) = \frac{1}{1+\ln 2}$

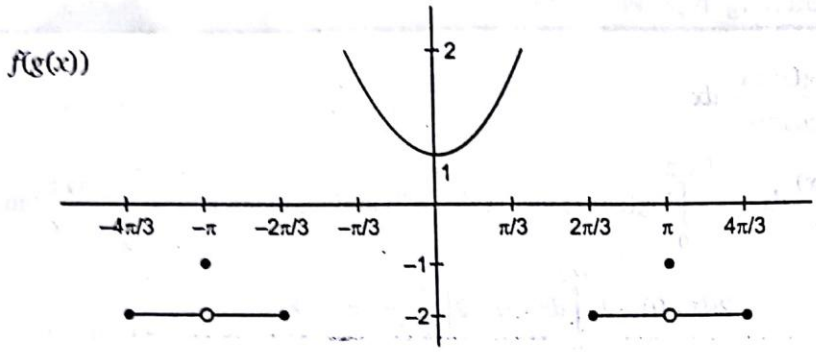
$2(f^{-1})'(\ln 4) = \frac{2}{1+\ln 2} > 0, 1$

(D) $l = \lim_{x\to\infty} (x\ln x)^{\frac{1}{x^2+1}}$

$$\ln l = \lim_{x\to\infty} \frac{\ln x + \ln(\ln x)}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{\ln x}\cdot\frac{1}{x}}{2x} = 0$$

$\ln(l) = 0 \Rightarrow l = 1$

2. $g(f(x))$ →
- $\sec 2, \quad -2 \le x < -1$
- $\sec 1, \quad -1 \le x < 0$
- $\sec x, \quad 0 \le x \le 2$



3. (A) $f(1^+) = f(1^-) = -1$

(B) $\int\limits_2^3 ([x]\cdot\{x\} - |x|)\cdot dx = \int\limits_2^3 (2(x-2) - x)\, dx = \left(\frac{x^2}{2} - 4x\right)_2^3 = \frac{-3}{2}$

(C) $[x]\cdot\{x\} = -1 \qquad x \le 0$

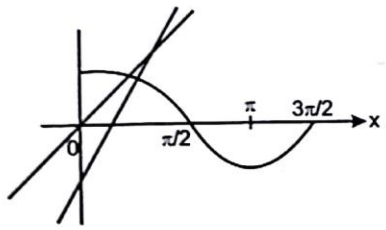
$x = -3 + \frac{1}{3}, -2 + \frac{1}{2}$

(D) $l = \lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} ([x]\{x\} - |x|) = -4$

4. (A) $\lim\limits_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}} = \frac{1}{2}$

(B) $\lim\limits_{x\to 0}\frac{\log_{\sec x/2}\cos x}{\log_{\sec x}\cos\frac{x}{2}} = \lim\limits_{x\to 0}(\log_{\sec\frac{x}{2}}\cos x)^2 = \lim\limits_{x\to 0}\left(\frac{\ln\cos x}{\ln\sec x/2}\right)^2 = 2$

(C)



(D) $\sin x \ne \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$

5. $f(1^+) = f(1^-) = f(1) \Rightarrow b = 0$

$f(3^-) = f(3^+) = f(3)$

$3 = 9p + 3q + 2 \Rightarrow 3p + q = 0$

$f'(x) = 2ax - a \qquad x < 1$

$\quad = 1 \qquad 1 \le x < 3$

$\quad = 2px + q \qquad x > 3$

$f'(3^+) = f'(3^-) = f'(3)$

$6p + q = 1 \Rightarrow p = \frac{1}{3}, q = -1$

$f'(1^+) \neq f'(1^-)$

$a \neq 1$

Exercise-5 : Subjective Type Problems

1. $f(x)$ is discontinuous at $x = 1, |f(x)|$ is diff. every where

$f(1) = -f(1^+) = -f(1^-)$

$\Rightarrow \quad 3 = -(0 + b)$

$\Rightarrow \quad b = -3$

$f'(1^+) = -f'(1^-)$ (as $|f(x)|$ is differentiable every where)

$\therefore \quad 1 = -(2a - a) \Rightarrow a = -1$

Continuous at $x = 3$,

So, $5 = 9p + 3q + 2$

$\Rightarrow \quad 3p + q = 1$ ...(1)

$f'(x)$ is continuous at $x = 3$

So, $f'(3^-) = f'(3^+)$

$\Rightarrow \quad 6p + q = 1$ ...(2)

On solving (1) & (2) we get, $p = 0, q = 1$

So, $|a + b + p + q| = |-1 - 3 + 0 + 1| = 3$

2. $\sin^{-1} y = 8 \sin^{-1} x$

$\frac{y'}{\sqrt{1 - y^2}} = \frac{8}{\sqrt{1 - x^2}}$

$(1 - x^2)(y')^2 = 64(1 - y^2)$

$(1 - x^2)y'' - xy' = -64y$

3. $yy' = 4a$

$(y')^2 + yy'' = 0$

4. $f(x)$ is discontinuous at $x = -\sqrt{3}, -\sqrt{2}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$

$\sin \pi x = 0$ at $x = -2, -1, 0, 1, 2, 3$

So, continuous at these points.

5. Let $f'(x) = K$

$\Rightarrow \quad f(x) = Kx + c$

$\Rightarrow \quad f(9) - f(-3) = 12K$

Maximum value of $f(9) - f(-3) = 96$

6. $g(x) = \sin x^3 - x^3 + 1 \qquad x \ge 1$

$= \sin x^3 + x^3 - 1 \qquad 0 \le x < 1$

$= -\sin x^3 - x^3 - 1 \qquad -1 \le x < 0$

$= -\sin x^3 + x^3 + 1 \qquad x \le -1$

Function is not differentiable at $x = -1, 1$

7. $F(x) = g(x) \qquad x > 1$

$= \dfrac{f(x) + g(x)}{2} \qquad x = 1$

$= f(x) \qquad -1 < x < 1$

$= \dfrac{f(x) + g(x)}{2} \qquad x = -1$

$= g(x) \qquad x < -1$

If $F(x)$ is continuous at $x = 1$

$F(1^+) = F(1) = F(1^-)$

$b = a + 3$

If $F(x)$ is continuous at $x = -1$

$F(-1^-) = F(-1) = F(-1^+)$

$a + b = 5$

8. $f^{-1}(x) = 2 - x \qquad 2 \le x \le 5$

$= 2 + x \qquad -2 < x < 2$

9. $f(x) + 2f(1-x) = x^2 + 2$

$f(1-x) + 2f(x) = (1-x)^2 + 2 \Rightarrow f(x) = \dfrac{(x-2)^2}{3}$

10. $g(x) = x(x-3)(x-7)$

$f(g(x)) = \text{sgn}(x(x-3)(x-7))$

11. $\dfrac{d^2}{dx^2}(\sin^2 x - \sin x + 1) = -4\sin^2 x + \sin x + 2$

12. $f(x) = a\cos(\pi x) + b$

$f'(x) = -a\pi \sin(\pi x)$

$\displaystyle\int_{1/2}^{3/2} f(x)\,dx = -\frac{2a}{\pi} + b = \frac{2}{\pi} + 1 \Rightarrow a = -1,\ b = 1$

13. $\alpha'(x) = f'(x) - 2f'(2x)\ ; \quad \beta'(x) = f'(x) - 4f'(4x)$

$\alpha'(1) = f'(1) - 2f'(2) = 5$

$\alpha'(2) = f'(2) - 2f'(4) = 7$

$\beta'(1) = f'(1) - 4f'(4) = \alpha'(1) + 2\alpha'(2) = 5 + (2 \times 7) = 19$

$\beta'(1) - 10 = 19 - 10 = 9$

14. $g(f(x)) = x$

$g'(f(x))f'(x) = 1$

$f(1) = -7/6$

$\therefore\ x = 1$

$$g'\left(-\frac{7}{6}\right)f'(1) = 1$$

$$g'\left(-\frac{7}{6}\right) = \frac{1}{f'(1)}$$

$$f'(x) = -4 \cdot e^{\frac{1-x}{2}}\left(-\frac{1}{2}\right) + x^2 + x + 1$$

$f'(1) = 2 + 1 + 1 + 1 = 5$

$$h(x) = ax^{-\frac{5}{4}} + bx^{\frac{1}{4}}$$

$$h'(x) = -\frac{5a}{4}x^{\frac{-9}{4}} + \frac{b}{4}x^{\frac{-3}{4}}$$

$$h'(5) = 0 \Rightarrow -\frac{5a}{4} \cdot 5^{-9/4} + \frac{b}{4} \cdot 5^{-3/4} = 0$$

$\Rightarrow\ 5a \cdot 5^{-3/2} = b$

$\Rightarrow\ \dfrac{a}{b} = 5^{1/2}$

$$\left(\frac{a}{b}\right)^2 = 5$$

$$\frac{a^2}{5b^2 g'\left(\frac{-7}{6}\right)} = \frac{5}{5 \times \frac{1}{5}} = 5$$

16. Let $$\lim_{x \to \infty}\left(f(x) + \int_0^x f(t)\,dt\right) = l \quad \text{...(1)}$$

$$\lim_{x \to \infty} \frac{\left(e^x \int_0^x f(t)\,dt\right)}{(e^x)} = l$$

$$\Rightarrow \quad \lim_{x\to\infty} \frac{e^x \int_0^x f(t)\,dt}{e^x} = l$$

$$\Rightarrow \quad \lim_{x\to\infty} \int_0^x f(t)\,dt = l \qquad \ldots(2)$$

From (1) and (2) we get, $\lim_{x\to\infty} f(x) = 0$

17. $f(0) = 0, f'(0) = 1, f''(0) = 1, f'''(0) = 2$

$g(f(x)) = x \quad \Rightarrow \quad g'(f(x))f'(x) = 1$

$$\Rightarrow \quad g''(f(x)) = \frac{-f''(x)}{(f'(x))^3}$$

$$\Rightarrow \quad g'''(f(x))f'(x) = -\left[\frac{(f'(x))^3 \cdot f'''(x) - 3(f''(x))^2(f'(x))^2}{(f'(x))^6}\right]$$

Put $x = 0$

$$g'''(0)\,1 = -\left[\frac{1\times 2 - 3\times 1}{1}\right] = 1$$

19. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{f(x)}{1+h/x} + \frac{f\left(1+\frac{h}{x}\right)}{x} - f(x)}{h} = \lim_{h\to 0} \frac{f(x)\left(-\frac{h}{x}\right)}{h\left(1+\frac{h}{x}\right)} + \frac{f\left(1+\frac{h}{x}\right)}{hx}$$

$$= \frac{-f(x)}{x} + \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{x^2\left(\frac{h}{x}\right)} \qquad \text{(as } f(1) = 0\text{)}$$

$$f'(x) = \frac{-f(x)}{x} + \frac{f'(1)}{x^2}$$

$$xf'(x) + f(x) = \frac{1}{x}$$

$$\frac{d}{dx}(xf(x)) = \frac{1}{x}$$

$$xf(x) = \int \frac{1}{x}\,dx$$

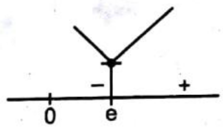
$$xf(x) = \ln x + k$$

Put $x = 1$, we get $k = 0$

$\therefore \quad f(x) = \dfrac{\ln x}{x}$

$H(x) = \dfrac{1}{f(x)} = \dfrac{x}{\ln x}$

$H'(x) = \dfrac{\ln x \cdot 1 - 1}{(\ln x)^2} \qquad H(x) \geq e$



$H(e) = e$

$\therefore \quad \lim_{x \to e} \left[\dfrac{1}{f(x)}\right] = 2$

21. $f'(x) = \tan^{-1}(x^2) + \dfrac{2x^2}{1+x^4} + 4x^3$

$f''(x) = \dfrac{2x}{1+x^4} + 2\left(\dfrac{(1+x^4)\cdot 2x - x^2(4x^3)}{(1+x^4)^2}\right) + 12x^2$

22. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = -3\sin\theta\cos\theta = -\dfrac{3}{2}\sin 2\theta$

$\dfrac{d^2y}{dx^2} = \dfrac{-3\cos 2\theta}{\sin\theta}$

23. Let $8x - 16 = t^2 \Rightarrow \sqrt{\dfrac{t^2+16+8t}{8}} + \sqrt{\dfrac{t^2+16-8t}{8}} = \dfrac{|t+4|+|t-4|}{2\sqrt{2}}$

24. $f(x) = [x] \qquad 0 < x < 1$

$= \{x\} \qquad 1 \leq x < \dfrac{5}{4}$

$= \left|x - \dfrac{3}{2}\right| \qquad \dfrac{5}{4} \leq x < 2$

No. of points where $f(x)$ is non-differentiable are three.

$x = 1, \dfrac{5}{4}, \dfrac{3}{2}$

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Chapter 4 – Application of Derivatives

4 APPLICATION OF DERIVATIVES

Exercise-1 : Single Choice Problems

1. Maximum value of $f(x) = 3$

Minimum value of $f(x) = -1$

2. $f''(x) = 6x - 6$

$f'(x) = 3x^2 - 6x + 3 \qquad (\because f'(2) = 3)$

$f(x) = x^3 - 3x^2 + 3x - 1 \qquad (\because f(2) = 1)$

5. $V = \frac{4}{3}\pi(10 + T)^3 - \frac{4}{3}\pi(10)^3$

$\frac{dV}{dt} = 4\pi(10 + T)^2 \frac{dT}{dt}$

$\Rightarrow \quad \frac{dT}{dt} = \frac{1}{18\pi} \qquad (\because T = 5\text{ cm})$

6. $g(x) = \frac{(|x| - 1)(|x| - 2)}{(|x| - 3)(|x| - 4)}$

$g(x)$ is an even function so there is an extrema at $x = 0$.

Also number of extrema for $x > 0$ will be equal to number of extrema for $x < 0$

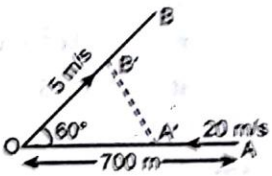
for $x > 0$

$$g(x) = \frac{(x - 1)(x - 2)}{(x - 3)(x - 4)}$$

Number of extrema $= 2$

$\Rightarrow \quad$ Total extrema $= 5$

7. $A'B' = \sqrt{\left(700 - \frac{45}{2}t\right)^2 + \frac{75}{4}t^2}$



$(A'B')_{min}$ at $t = 30$ sec

8. $f(0^-) \geq f(0) \Rightarrow a \geq 3$

9. $f(x) \begin{cases} 3+k-x, & x \leq k \\ a^2-2+\frac{\sin(x-k)}{x-k}, & x > k \end{cases} \Rightarrow f'(k^+) > f(k), f'(k^-) > f(k)$

So, $\lim_{x\to k^+} (a^2-2) + \frac{\sin(x-k)}{(x-k)} = a^2 - 1 > 3$

$a^2 > 4$

$|a| > 2$

10. $\frac{dy}{dx} = 3x^2 - 4x + C_1$

$y = x^3 - 2x^2 + C_1x + C_2$

Also, $\left.\frac{dy}{dx}\right]_{at\ x=1} = 0$ and $y]_{at\ x=1} = 5$

11. $m_1 = \left.\frac{dy}{dx}\right]_{at\ (1,\ 2)} = 2a + b$

$m_2 = g'(x) = \left.\frac{dy}{dx}\right]_{at\ (-2,\ 2)} = 2 \Rightarrow 2a + b = -\frac{1}{2}$

Also, $2 = a + b + \frac{7}{2}$

12. $18y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{18y}$

$\Rightarrow \frac{a^2}{6b} = 1 \Rightarrow a^2 = 6b$

Also, $9b^2 = a^2$

13. $\frac{dy}{dx} = 3x^2 - 4x + c$

at $x = 1, \frac{dy}{dx} = 0 \quad \Rightarrow c = 1$

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

$$y = x^3 - 2x^2 + x + d$$

at $x = 1, y = 5 \quad \Rightarrow 5 = 1 - 2 + 1 + d$

$\Rightarrow \quad d = 5$

14. $A(0,2)$

$$5\alpha^2(3x^2) + 10\alpha(2x) + 1 + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-15\alpha^2x^2 - 20\alpha x - 1}{2}$$

$\Rightarrow \frac{dy}{dx}$ at $A = -\frac{1}{2}$

Equation of normal at A is $y = 2x + 2$

Let normal meets the curve at B

$$5\alpha^2x^3 + 10\alpha x^2 + x + 4x + 4 - 4 = 0$$

$$5x(\alpha x + 1)^2 = 0$$

$$x = -\frac{1}{\alpha}$$

So, $B\left(\frac{-1}{\alpha}, \frac{-2}{\alpha} + 2\right)$

$\therefore$ Slope of tangent at $B = \frac{-15 + 20 - 1}{2} = 2$

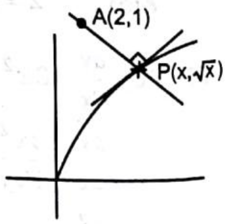
15. $f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$

$f'(x) = -\sin x - \sin 2x + \sin 3x = 2\sin x(2\cos x + 1)(\cos x - 1) = 0$

16. Closest distance exist always alone the normal

$$\therefore \quad \frac{1-\sqrt{x}}{2-x} \times \frac{dy}{dx} = -1$$

$$\frac{1-\sqrt{x}}{2-x} \times \frac{1}{2\sqrt{x}} = -1$$



Let $\sqrt{x} = t$

$$x = \frac{2+\sqrt{3}}{2}$$

17. Let $x = 2\sin\theta$

$$y = \ln\left(\frac{2 + 2\cos\theta}{2 - 2\cos\theta}\right) - 2\cos\theta = \ln\left(\frac{2\cos^2\theta/2}{2\sin^2\theta/2}\right) - 2\cos\theta$$

$$= 2\ln\left(\cot\frac{\theta}{2}\right) - 2\cos\theta$$

$$\frac{dy}{d\theta}=\frac{1}{\cot\theta/2}\left(-\text{cosec}^2\frac{\theta}{2}\right)+2\sin\theta$$

$$\frac{dy}{d\theta}=\frac{-2}{\sin\theta}+2\sin\theta=\frac{-2\cos^2\theta}{\sin\theta}$$

$$\frac{dx}{d\theta}=2\cos\theta;\quad \frac{dy}{dx}=-\cot\theta$$

$$\left(y-2\ln\left(\cot\frac{\theta}{2}\right)+2\cos\theta\right)=-\cot\theta(x-2\sin\theta)$$

$$\therefore\quad T=\left(0,2\ln\left(\cot\frac{\theta}{2}\right)-2\cos\theta+2\cos\theta\right)$$

$$P=\left(2\sin\theta,2\ln\cot\frac{\theta}{2}-2\cos\theta\right)$$

$$PT^2=(\sqrt{4\sin^2\theta+4\cos^2\theta})=4$$

18. $g'(x)=(2x^2-\ln x)f(x)$

$$f'(x)=\frac{1}{\ln x^3}3x^2-\frac{1}{\ln x^2}2x$$

$$f'(x)=\frac{x^2-x}{\ln x}$$

$$f'(x)=\frac{x(x-1)}{\ln x}>0\ \forall\ x>1;\quad f(x)>f(1)\Rightarrow f(x)>0\ \forall\ x>1$$

For $g(x)$ is increasing

$g'(x)>0\Rightarrow 2x^2-\ln x>0$ as $(f(x)>0)$

Let $H(x)=2x^2-\ln x$

$$H'(x)=4x-\frac{1}{x}=\frac{4x^2-1}{x}>0 \text{ when } x>1$$

$H(x)>H(1)\Rightarrow H(x)>2$

$\therefore\quad g'(x)>0\ \forall\ x\in(1,\infty)$

$\therefore\quad g(x)$ is increasing on $(1,\infty)$.

19. $f'(x)=3x^2+12x+a$

$f'(x)<0$ in $(-3,-1)$

Product of the roots $=\dfrac{a}{3}=3\Rightarrow a=9$

20. $f(x)=\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

$$f'(x)=\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\left(\frac{-2}{(1+x)^2}\right)=\frac{2}{2(1+x^2)}=\frac{1}{1+x^2}>0$$

$f'(x)$ is decreasing $\forall\ x \in R$

So, in $[0,1]$ $f(0) = \tan^{-1}(1) = \frac{\pi}{4}$ (max)

$f(1) = 0$ (min)

21. $f'(x) = 3x^2 + 2(a+2)x + 3a$

$\therefore$ $D \le 0$

$a^2 - 5a + 4 \le 0$

$\therefore$ $a \in [1, 4]$

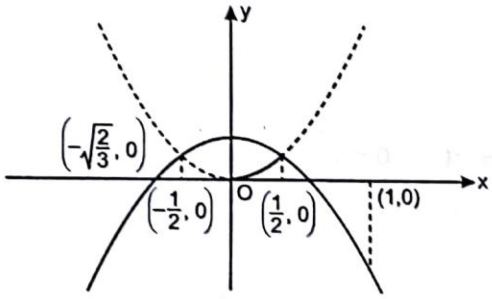
22. $f'(x) = 0$

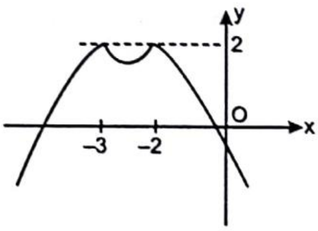
$\cos^2 x - \sqrt[3]{x} + x^{1/3} - \frac{1}{2} = 0$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm\frac{1}{\sqrt{2}}$

$\therefore$ total number is 12.

23. 

24. 

$b^2 + 1 \ge 2$

25. $f(x)$ is continuous and differentiable in $[-1, 1]$.

26. $\frac{\cos x_1}{x_1} = -\sin x_1 \Rightarrow x_1 = -\cot x_1$

Point $(x_1, \cos x_1)$ always lie on $\frac{1}{y^2} = \frac{1}{x^2} + 1$

27. $x + \frac{a}{x^2} > 2 \; \forall \; x \in (0, \infty)$

$f(x) = x^3 - 2x^2 + a > 0 \; \forall \; x \in (0, \infty)$

$f'(x) = 3x^2 - 4x = 3x\left(x - \frac{4}{3}\right)$

Minimum value at $x = \frac{4}{3}$

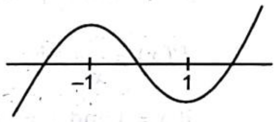
$\frac{64}{27} - 2\left(\frac{16}{9}\right) + a > 0 \;\; \Rightarrow a > \frac{32}{27}$

29. $f'(x) = \cos^2 x + \cos x + 2 > 0$

$f(x)_{\min} = f(0) = 0$

$f(x)_{\max} = f(2\pi) = 5\pi$

31. $f(x) = x^3 - 3x + c = 0$



$f'(x) = 3(x^2 - 1)$

$\Rightarrow \;\; f(1) f(-1) < 0$

$(c-2)(2+c) < 0$

32. $f'(x) = e^x (x-1)(x-2) < 0$

33. $\frac{dy}{dx} = 3ax^2 + 2bx + c = 0$ has one root $\Rightarrow D = b^2 - 3ac = 0 \Rightarrow b^2 = 6$

34. Let $x = \tan\theta$ then $y = \cos^2\theta$

$\left|\frac{dy}{dx}\right| = |2\sin\theta\cos^3\theta|$

$\left|\frac{dy}{dx}\right|_{\max}$ at $\theta = \frac{\pi}{6}$

35. $h(x) = f(x) - g(x) = 2x - 3\sin x + x\cos x$

$h(0) = 0$

$h'(x) = 2 - 2\cos x - x\sin x$

$h'(0) = 0$

$h''(x) = \sin x - x\cos x$

$h''(0) = 0$

$h'''(x) = x\sin x > 0 \; \forall \; x \in \left(0, \frac{\pi}{2}\right)$

36. $f(x) = 2\tan^{-1}(g(x)) \qquad |g(x)| \le 1$

$= \pi - 2\tan^{-1} g(x) \qquad g(x) > 1$

$= -\pi - 2\tan^{-1} g(x) \qquad g(x) < -1$

$$f'(x) = \frac{2g'(x)}{1+(g(x))^2} \qquad |g(x)|<1$$

$$= -\frac{2g'(x)}{1+(g(x))^2} \qquad |g(x)|>1$$

37. $\lim_{x\to e^a}\left[\frac{7}{3}\left[\frac{\ln(1+7f(x))}{7f(x)}\right]-\frac{1}{3}\left(\frac{\sin f(x)}{f(x)}\right)\right]=2$

38. If $f(x)$ is strictly decreasing for all x,

$f'(x) = \log_{1/3}(\log_3(\sin x + a)) \le 0$

$\Rightarrow \quad \sin x + a \ge 3 \ \forall \ x \in R$

$\Rightarrow \quad a \ge 4$

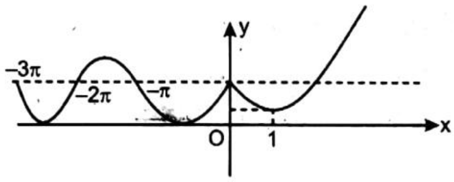
39. $f(x) = a\ln|x| + bx^2 + x$

$f'(x) = \frac{a}{x} + 2bx + 1 = \frac{2bx^2 + x + a}{x}$

if $x = 1$ and $x = 3$ are point of extrema.

$\Rightarrow \quad -\frac{1}{2b} = 4$ and $\frac{a}{2b} = 3$

40.



$f(x)$ has local maximum at $x = 0$.

41. $f(x) = \int_1^x (t-a)^{2n}(t-b)^{2m+1}\,dt$

$f'(x) = (x-a)^{2n}(x-b)^{2m+1}$

No sign change of $f'(x)$ about $x = a$.

$f'(x)$ will change sign from negative to positive at $x = b \Rightarrow$ Point of minima.

43. Let point P on the curve $y^2 = x^3$ is $P(t_1^2, t_1^3)$.

Equation of tangent at $P(t_1^2, t_1^3)$ is

$$y - t_1^3 = \frac{3}{2}t_1(x - t_1^2)$$

If this intersect the curve again at $Q(t_2^2, t_2^3)$

$\Rightarrow \quad t_2 = -\frac{t_1}{2}$

$$\frac{\tan\alpha}{\tan\beta} = \frac{(3t_1/2)}{(3t_2/2)} = -2$$

44. $y^2 = \alpha x^3 - \beta$

if $(2, 3)$ is lie on the curve

$$8\alpha - \beta = 9 \quad \text{...(1)}$$

Slope of normal at $(2, 3)$

$$-\frac{1}{4} = -\frac{1}{2\alpha} \Rightarrow \alpha = 2$$

45. Equation of tangent at $(0, 1)$ to the curve $y - 1 = kx$ meet x-axis at $(a, 0)$ then

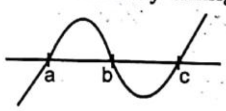
$$-2 \le -\frac{1}{k} \le -1 \Rightarrow k \in \left[\frac{1}{2}, 1\right]$$

46. $f(x) = \int_0^{\sqrt{x}} e^{\frac{-u^2}{x}} du = \sqrt{x}\int_0^1 e^{-t^2} dt$

where $t = \frac{u}{\sqrt{x}}$

$\Rightarrow f(x) = K\sqrt{x}, \quad K > 0$

47. $f''(\alpha) = 0 \Rightarrow x = \alpha$ is the point where concavity changes.



48. $f(x) = x^6 - x - 1$

$f'(x) = 6x^5 - 1 > 0 \ \forall\ x \in [1, 2]$

If $f(1) = -1 < 0$ and $f(2) = 2^6 - 3 > 0$ then $f(x)$ has one root in $[1, 2]$.

49. Every line passing from (a, b) is normal to the circle $(x - a)^2 + (y - b)^2 = k$

50. $f'(x) = \cos x(3\sin^2 x - m) = 0$

$\sin^2 x = \frac{m}{3} \Rightarrow 0 < \frac{m}{3} < 1$

$$0 < m < 3$$

51. Let $y = x^{1/x}$

$y' = x^{(1/x)-2}(1 - \ln x)$

$f(x)$ is increasing $\quad (0, e)$

and $f(x)$ is decreasing $\quad (e, \infty)$

52. Let $y = mx$

Point of tangency be (x_1, y_1)

$\Rightarrow \quad mx_1 = x_1^3 + x_1 + 16 \;\&\; m = 3x_1^2 + 1$

$\Rightarrow \quad x_1(3x_1^2 + 1) = x_1^3 + x_1 + 16$

$x_1 = 2$

$m = 13$

53. $y' = 3x^2 - 6x + 6$

$y'' = 6x - 6 = 0$

$x = 1$

$y' = 3$

54. Let $H(x) = \ln(f(x) + f'(x) + \ldots f^n(x)) - x$

$\Rightarrow \quad H(a) = H(b)$

$\Rightarrow \quad H'(c) = 0$ (by L.M.V.T.)

$\Rightarrow \quad \dfrac{f'(c) + f''(c) + \ldots + f^{n+1}(c)}{f(c) + f'(c) + \ldots + f^n(c)} - 1 = 0$

$\Rightarrow \quad f^{n+1}(c) = f(c)$

55. $h(x) = g(x) + x$

$\Rightarrow \quad h'(x) = g'(x) + 1$

$\Rightarrow \quad g'(x) = h'(x) - 1$

$\Rightarrow \quad g''(x) = h''(x)$

$\Rightarrow \quad h''(x) - 3(h'(x) - 1) > 3$

$\Rightarrow \quad h''(x) - 3h'(x) > 0$

$\Rightarrow \quad \dfrac{d}{dx}(e^{-3x}h'(x)) > 0$

Let $P(x) = e^{-3x}h'(x)$

$\Rightarrow \quad P'(x) > 0$

$\Rightarrow \quad P(x)$ is an increasing function.

$P(0) = h'(0) = 0$

$\Rightarrow \quad P(x) > 0 \;\forall\; x > 0$

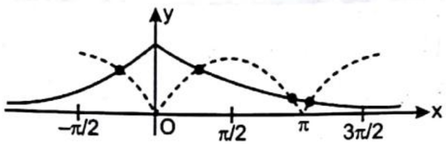
$\Rightarrow \quad h'(x) > 0 \;\forall\; x > 0$

$\Rightarrow \quad h(x)$ is an increasing function $\forall\; x > 0$

56. $\dfrac{dy}{dx} = -\dfrac{c}{(x+1)^2} = -1 \Rightarrow (x+1)^2 = c$

Point $(\sqrt{c} - 1, \sqrt{c})$ lie on the line $x + y = 3 \Rightarrow \sqrt{c} = 2$

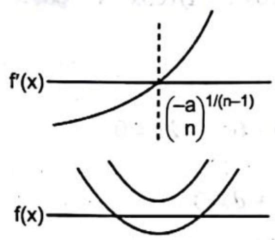
57. $|\sin x| = e^{-x^2}$



60. $x^n + ax + b = 0$

x is even.

$nx^{n-1} + a = f'(x)$



61. $f(b) = \left|\sin x + \dfrac{2}{3+\sin x} + b\right|_{\max} \quad \forall\ x \in R$

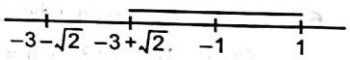
$\sin x = t$

$g(t) = t + \dfrac{2}{3+t} \qquad t \in [-1, 1]$

$g'(t) = 1 - \dfrac{2}{(3+t)^2} > 0$

$(3+t)^2 - 2 > 0$

$(3 + t - \sqrt{2})(3 + t + \sqrt{2}) > 0$

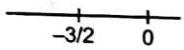


$g(t) = t + \dfrac{2}{3+1}$ increasing $\forall \in [-1, 1]$

$g(t)_{\max} = \dfrac{3}{2}$

$g(t)_{\min} = 0$

$f(b) = \begin{vmatrix} \dfrac{3}{2} + b & \text{if } b \geq -\dfrac{3}{4} \\ -b & b < -\dfrac{3}{4} \end{vmatrix}$



min. of $f(b) = -\left(\dfrac{-3}{4}\right) = \dfrac{3}{4}$

62. $y = \dfrac{x}{1+x^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1-x^2}{(1+x^2)^2}$

63. $f^{-1}(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

$$\frac{d}{dx}f^{-1}(x) = \frac{-2}{\sqrt{1-\frac{x^2}{9}}}\left(\frac{1}{3}\right)$$

64. $f(x) = \sin x + \tan x - 2x$

$f'(x) = \cos x + \sec^2 x - 2 = 0$

$\cos^3 x - 2\cos^2 x + 1 = 0 \Rightarrow (\cos x - 1)(\cos^2 x - \cos x - 1) = 0$

$\cos x = 1, \frac{1-\sqrt{5}}{2}$

65. $\frac{a+2c}{b+3b} + \frac{4}{3} = 0 \quad \Leftrightarrow \quad 3a + 4b + 6c + 12b = 0$

$\Leftrightarrow \quad \frac{1}{4}a + \frac{b}{3} + d = 0$

Consider $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$

then $\quad f(0) = 0 = f(1)$

$\therefore \quad f(x)$ satisfies the conditions of Rolle's theorem in [0, 1].

Hence, $f'(x) = 0$ has at least one solution in (0, 1).

66. $f'(x) = \phi(x)\cdot(x-2)^2$

$\phi(2) > 0 \Rightarrow f'(x) > 0 \Rightarrow f(x)\uparrow$

$\phi(2) < 0 \Rightarrow f'(x) < 0 \Rightarrow f(x)\downarrow$

67. $f(1) = f(3) \Rightarrow a + b - 5 = 3a + b - 27 \Rightarrow a = 11$

$f'(c) = 3c^2 - 12c + a = 0 \Rightarrow b \in R$

70. Let $x = \frac{3at}{2^{2/3}}, y = at^{3/2}$

$$\frac{dy}{dx} = \frac{2\left(\frac{9a^2t^2}{2^{4/3}}\right)}{9a^2t^{3/2}} = -\cot\alpha \Rightarrow \sqrt{t} = -2^{1/3}\cot\alpha$$

and $P = \cos\alpha\left(\frac{3at}{2^{2/3}} - \frac{at^{3/2}}{-\cot\alpha}\right)$

$\Rightarrow \quad \frac{P}{a} = \cos\alpha\cot^2\alpha$

Exercise-2 : One or More than One Answer is/are Correct

1. Equation of tangent to $y = x^3$

$$y - x_1^3 = 3x_1^2(x - x_1)$$

Equation of tangent to $y = x^{1/3}$ is

$$y - x_2^{1/3} = \frac{1}{3x_2^{2/3}}(x - x_2)$$

If these tangents represent same line

$$\frac{1}{1} = \frac{9x_1^2 x_2^{2/3}}{1} = \frac{-2x_1^3}{\frac{2}{3}x_2^{1/3}} \Rightarrow x_1 = \pm\frac{1}{\sqrt{3}}$$

2. (a) $f'(C_1) = \frac{f(4) - f(0)}{4 - 0} = \frac{1}{4}$; $C_1 \in (0, 4)$

(c) $f'(C_1) = \frac{f(8) - f(0)}{8 - 0} = \frac{1}{8}$; $C_1 \in (0, 8)$

$f(C_2) = f(8) = 1$

(d) Let $g(x) = \int_0^{x^3} f(t)\,dt$

$$\Rightarrow \quad g(0) = 0,\ g(2) = \int_0^8 f(t)\,dt$$

$$g'(\alpha) = 3\alpha^2 f(\alpha^3) = \frac{g(2) - g(0)}{2} \quad \alpha \in (0, 2)$$

and $$g'(\beta) = 3\beta^2 f(\beta^3) = \frac{g(2) - g(0)}{2} \quad \beta \in (0, 2)$$

$$g'(\alpha) + g'(\beta) = g(2) - g(0) = \int_0^8 f(t)\,dt$$

4. $f(x) = 2x^4 + x^4 \sin\frac{1}{x} \qquad x \neq 0$

$= 0 \qquad x = 0$

$$f'(x) = 8x^3 + 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$$

5. $-1 \le f''(x) \le 1$

$-x \le f'(x) \le x \qquad (\because f'(0) = 0)$

$-\frac{x^2}{2} \le f(x) < \frac{x^2}{2} \qquad (\because f(0) = 0)$

6. $f''(x) > 0 \ \forall \ x \in [-3, 4]$

$\Rightarrow \quad f'(x)$ is increasing for $x \in [-3, 4]$

7. $f''(x) > 0 \ \forall \ x \in [0, 2]$

$\Rightarrow \quad f'(x) \uparrow$

$f'(C_1) = \dfrac{f(1) - f(0)}{1 - 0}, C_1 \in (0, 1)$ and $f'(C_2) = \dfrac{f(2) - f(1)}{2 - 1}, C_2 \in (1, 2)$

$f'(C_1) < f'(C_2) \Rightarrow f(0) + f(2) > 2f(1)$

Similarly applying LMVT between $\left[0, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 2\right]$

$$\frac{f(2) - f\left(\frac{2}{3}\right)}{\frac{4}{3}} > \frac{f\left(\frac{2}{3}\right) - f(0)}{\frac{2}{3}} \Rightarrow 2f(0) + f(2) > 3f\left(\frac{2}{3}\right)$$

8. Let $g''(x) = a(x - 1)$

$g'(x) = \dfrac{ax^2}{2} - ax + b$

$g'(-1) = 0 \Rightarrow b = -\dfrac{3a}{2}$

$g(x) = \dfrac{ax^3}{6} - \dfrac{ax^2}{2} + bx + c \Rightarrow g(x) = x^3 - 3x^2 - 9x + 5 \qquad (\because g(-1) = 10, g(3) = -22)$

9. $f(x) = 2x^3 - 3(\lambda + 2)x^2 + 2\lambda x + 5$

$f'(x) = 6x^2 - 6(\lambda + 2)x + 2\lambda = 0$ has two real roots, then

$D > 0 \Rightarrow 3\lambda^2 + 8\lambda + 12 > 0$

$\Rightarrow \quad \lambda \in R$

10. $f(x) = 1 + x\ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2}$

$f'(x) = \ln(x + \sqrt{1 + x^2})$

$f'(x) \geq 0$ for $\forall \ x \in [0, \infty)$

$f'(x) \leq 0$ for $\forall \ x \in (-\infty, 0]$

11. $f(x, y) = x^m (k - x)^n$

$f'(x, y) = mx^{m-1}(k - x)^n - x^m n \cdot (k - x)^{n-1} = 0$

$\Rightarrow \quad x = \dfrac{mk}{m + n}$

Maximum value $= \dfrac{k^{m+n} \cdot m^m \cdot n^n}{(m + n)^{m+n}}$

12. Let line is tangent at $(3t_1^2, 2t_1^3)$ and normal at $(3t_2^2, 2t_2^3)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{3t_1^2, 2t_1^3} = t_1$

So, slope of normal at $(3t_2^2, 2t_2^3) = -\frac{1}{t_2}$

$\Rightarrow t_1 = -\frac{1}{t_2}$

$t_2 = -\frac{1}{t_1}$

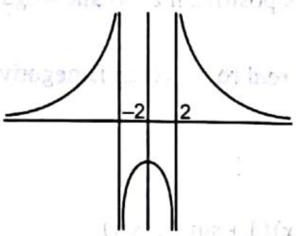
$\Rightarrow t_1 = \frac{2t_1^3 + \frac{2}{t_1^3}}{3\left(t_1^2 - \frac{1}{t_1^2}\right)}$

$\Rightarrow t_1^2(t_1^4 - 3) = 2$ or $t_1 = \pm\sqrt{2}$

13. $m = \frac{dy}{dx}$

$\frac{|my|}{x+y} = \frac{y}{x}$ then solve it.

14. First draw the graph $f(x) = \frac{1}{x^2 - 4}$



While drawing diff. possibilities of $y = ax^2 + bx + c$

We get possible intersections.

15. $y' = 3x_1^2$

$3x_1^2 = \frac{x_1^3 - 8}{x_1 - 2} \Rightarrow x_1 = -1$ or 2

$y' = 3$ or $y' = 12$

16. Let $f(x) = x + \cos x - a$

$f'(x) = 1 - \sin x \geq 0 \ \forall \ x \in R$

$\Rightarrow f(x)$ is increasing.

$\Rightarrow x + \cos x - a = 0$ for one positive value of $x, a \in (1, \infty)$

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17. Let $y = ax + b = f(x) \quad a \neq 0$

$f^{-1}(x) = \dfrac{x-b}{a}$

(1) $m_1 = a, m_2 = \dfrac{1}{a} \qquad \Rightarrow m_1 m_2 = 1$

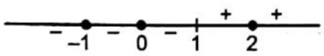
(2) $m_1 = a, m_2 = \dfrac{-1}{a} \qquad \Rightarrow m_1 m_2 = -1$

(3) $m_1 = -a, m_2 = \dfrac{1}{a} \qquad \Rightarrow m_1 m_2 = -1$

(4) $m_1 = -a, m_2 = \dfrac{-1}{a} \qquad \Rightarrow m_1 m_2 = 1$

18. $f'(x) = e^x(x^2-1)x^2(x+1)^{2011}(x-2)^{2012}$

$= e^x x^2 (x+1)^{2012}(x-1)(x-2)^{2012}$



$x = -1, 0, 2$ are points of inflections and might be more points in (1, 2).

$x = 1$ is point of minima (Answer can be given either d, ad, bd or abd)

19. $f(x) = \sin x + ax + b$

$f'(x) = \cos x + a$

if $a > 1$ then $f(x)$ is increasing.

So, only one real root, which is positive if $b > 0$ and negative if $b > 0$

if $a < -1$

$f(x)$ is decreasing so only one real root, which is negative if $b < 0$.

20. $f''(c) = 0 \quad$ for $\quad c \in (0, 1)$

$f''(x) > 0 \quad$ for $\quad x \in (0, c)$

$f''(x) < 0 \quad$ for $\quad x \in (c, 1)$

21. $f'(x) = 5\sin x \cos x(\sin x - \cos x)(1 + \sin x \cos x)$

Clearly, $\quad f'(x) > 0 \ \forall \ x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$f'(x) < 0 \ \forall \ x \in \left(0, \dfrac{\pi}{4}\right)$

$f(0) = 0, f\left(\dfrac{\pi}{2}\right) = 0$

By Rolle's theorem $\exists \ c \in \left(0, \dfrac{\pi}{2}\right) \Rightarrow f'(c) = 0$

Clearly, $\quad f(x) \geq f\left(\dfrac{\pi}{4}\right)$

$$f(x) \geq 2\left(\frac{1}{\sqrt{2}}\right)^5 - 1 = 2\left(\frac{1}{4\sqrt{2}}\right) - 1 = \frac{1}{2\sqrt{2}} - 1 \text{ and } f(x) < 0 \ \forall\ x \in \left(0, \frac{\pi}{2}\right)$$

22. $f(x) = x^{2\alpha+1} \ln x \qquad x > 0$

$\qquad = 0 \qquad x = 0$

$f(x)$ is not continuous at $\alpha = -\frac{1}{2}, -1$

23. $f'(x) = \dfrac{\cos x}{x}$

Clearly, $f'(x) > 0 \Rightarrow x \in \left(0, \frac{\pi}{2}\right) \cup \left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right) \forall\ n \in N$

and $f'(x) < 0 \Rightarrow x \in \left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right) \forall\ n \in N$

∴ $f(x)$ has a local minima at $x = (4n-1)\frac{\pi}{2} \ \forall\ n \in N$

and $f(x)$ has a local maxima at $x = \frac{\pi}{2}$ and $(4n+1)\frac{\pi}{2} \ \forall\ n \in N$

Also, $f''(x) = \dfrac{x(-\sin x) - \cos x}{x^2} = 0 \Rightarrow x \tan x + 1 = 0$

∴ All the points of inflection of $f(x)$ lie on the curve $x \tan x + 1 = 0$

Also, $f'(x) = 0 \Rightarrow x = (2n+1)\frac{\pi}{2} \ \forall\ n \in N$

∴ Number of values of x in $(0, 10\pi)$ in which $f'(x) = 0$ are 20.

24. $|f(x)| \leq 1$

Applying L.M.V.T. in $x \in (0, 1)$

$\Rightarrow \quad |f'(x)| = |f(1) - f(0)|$

$\quad |f(1) - f(0)| \leq 2$

$\Rightarrow \quad |f'(x)| \leq 2$ for atleast one x in $(0, 1)$

Similarly $|f'(x)| \leq 2$ for atleast one x is $(-1, 0)$

$\quad F(x) = (f(x))^2 + (f'(x))^2$

For atleast one x in $(0, 1)$ & $(-1, 0)$

$\quad |f'(x)| \leq 2$ & $|f(x)| \leq 1$

$\Rightarrow \quad (f'(x))^2 \leq 4$ & $(f(x))^2 \leq 1$

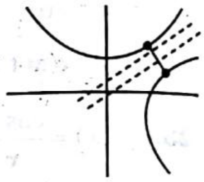
$\Rightarrow \quad (f'(x))^2 + (f(x))^2 \leq 5$

$\Rightarrow \quad F(x) \leq 5$, for atleast one x in $(-1, 0)$ & $(0, 1)$

25. $f\left(\frac{\pi}{2}\right) > f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi}{2}\right) > f\left(\frac{\pi^-}{2}\right)$

Also, absolute maximum occurs at $x = -1$

26. Symmetric about $y = x$



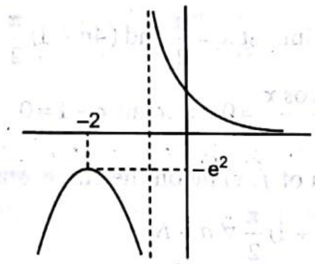
$$\frac{dy}{dx} = 1$$

$$2x = 1 \Rightarrow x = \frac{1}{2}$$

$$\text{Point} = \left(\frac{1}{2}, \frac{5}{4}\right)$$

27. $f'(x) > 0 \quad \Rightarrow \quad \frac{-(1+x)e^{-x} - e^{-x}}{(1+x)^2} > 0$

for $x < -2$ increasing.



28. Point (1, 2) lies on $y = mx + 5 \quad \Rightarrow m = -3$ ...(1)

Point (1, 2) lies on $x^3y^3 = ax^3 + by^3 \quad \Rightarrow 8b + a = 8$ ...(2)

$3x^2y^3 + x^3 \cdot 3y^2y^1 = 3ax^2 + 3by^2y^1 \Rightarrow a - 12b = -4$ ...(3)

$$\Rightarrow \quad a = \frac{16}{5},\ b = \frac{3}{5}$$

29. $\frac{f(x)-1}{f(x)+1} = \frac{x^4 + x^2 + 1}{(x^2+x+1)^2} = \frac{x^2 - x + 1}{x^2 + x + 1}$

$$\Rightarrow f(x) = \frac{x^2+1}{x}$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $f(x) = \dfrac{x+1}{x-1}$

$g(x) = \dfrac{x(x+1)}{x-1} = x + 2 + \dfrac{2}{x-1}$

$g'(x) = 1 - \dfrac{2}{(x-1)^2} = 0$

$x = 1+\sqrt{2}, 1-\sqrt{2}$

$g''(x) = \dfrac{4}{(x-1)^3}$

$g''(1+\sqrt{2}) > 0$

Minimum value of $g(x)$ is $3+2\sqrt{2}$.

2. $g'(x) = \dfrac{1}{2} \Rightarrow 1 - \dfrac{2}{(x-1)^2} = \dfrac{1}{2} \Rightarrow x = 3, -1$

Paragraph for Question Nos. 3 to 5

3. $g(1) = \int_0^1 f(t)\,dt = \int_0^1 (1-t)\,dt = t - \dfrac{t^2}{2}\Big|_0^1 = \dfrac{1}{2}$

4. For $x \in (2,3]$

$g(x) = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^x f(t)\,dt$

$g(x) = \dfrac{1}{2} + \dfrac{(x-2)^3}{3}$

at $x = \dfrac{5}{2}, g\left(\dfrac{5}{2}\right) = \dfrac{13}{24}$

$g'(x) = f(x)$

$g'\left(\dfrac{5}{2}\right) = \dfrac{1}{4}$

$y - \dfrac{13}{24} = \dfrac{1}{4}\left(x - \dfrac{5}{2}\right)$

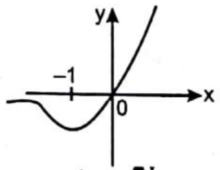
$12y = 3x - 1$

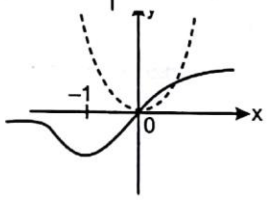
5. Slope of tangent at $P = \frac{1}{4}$

Slope of tangent at $R = \frac{2}{3}$

$$\tan\theta = \frac{5}{14}$$

Paragraph for Question Nos. 6 to 8

6. 

7. 

Paragraph for Question Nos. 9 to 11

9. By putting $x = 1, 2, -1, 0$ we get a, b, c, d clearly other roots product is 1.

10. $P(x) + k = 0$ has 4 distinct real roots.

$P(x) = -k$, where $-k \in (1, 2) \Rightarrow k \in (-2, -1)$

$\therefore$ pull the graph more than 1 and less than 2, now the graph intersect the x-axis in $(-2, -1), (-1, 0), (0, 1), (2, 3)$

$\therefore \quad -2 + (-1) + 0 + 2 = -1$

11. $P(x) = 0$ has two roots.

$P'(x) = 0$ has three root

$P(x) = 0$ has atleast 5 roots.

$(P'(x))^2 + P(x)P''(x) = 0 \Rightarrow (P'(x)P(x))' = 0$ has atleast four roots.

Paragraph for Question Nos. 12 to 14

Sol. The equation of chord AB will be $y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$

This line passes through $(0, 2x_1x_2)$

$$\therefore \quad 2x_1x_2 - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(-x_1)$$

$\Rightarrow \quad 2x_1x_2 = \dfrac{(x_2 - x_1)f(x_1) - x_1 f(x_2) + x_1 f(x_1)}{x_2 - x_1}$

$\Rightarrow \quad 2x_1x_2(x_2 - x_1) = x_2 f(x_1) - x_1 f(x_2)$

$\Rightarrow \quad \dfrac{f(x_1)}{x_1} - \dfrac{f(x_2)}{x_2} = 2(x_2 - x_1) \Rightarrow \dfrac{f(x_1)}{x_1} + 2x_1 = \dfrac{f(x_2)}{x_2} + 2x_2 = k$

$\therefore \quad \dfrac{f(x)}{x} + 2x = k$

$\therefore \quad f(x) = kx - 2x^2$

Given that $f(1) = -1$

$\therefore \quad -1 = k - 2 \Rightarrow k = 1$

$\therefore \quad f(x) = x - 2x^2$

12. $\therefore \quad \int_0^{1/2} f(x)\,dx = \left[\dfrac{x^2}{2} - \dfrac{2x^3}{3}\right]_0^{1/2} = \dfrac{1}{8} - \dfrac{2}{3}\cdot\dfrac{1}{8} = \dfrac{1}{8} - \dfrac{1}{12} = \dfrac{1}{24}$

13. $f'(x) = 1 - 4x \geq 0 \Rightarrow x \leq \dfrac{1}{4}$

14. $F(x) = f(x) + x = 2x - 2x^2$

Clearly, $F(0) = F(1) = 0$

$\therefore$ Rolle's theorem is applicable in $[0, 1]$.

Paragraph for Question Nos. 15 to 16

15. $f(x) = 1 + x\int_0^1 e^y f(y)\,dy + e^x \int_0^1 y\, f(y)\,dy$

Let $A = \int_0^1 e^y f(y)\,dy, \quad B = \int_0^1 y\, f(y)\,dy$

$\Rightarrow f(x) = 1 + Ax + Be^x \Rightarrow A = \int_0^1 e^x(1 + Ay + Be^y)\,dy$

$B = -\dfrac{2}{e+1}, A = -\dfrac{3}{2}$

$f'(x) + 3 > 0$

$\Rightarrow -\dfrac{3}{2} - \dfrac{2e^x}{(e+1)} + 3 > 0 \Rightarrow e^x < \dfrac{3(e+1)}{4} \Rightarrow \left[\dfrac{4}{3}e^x\right] = [e+1] = 3$

16. $Ax_1 + Be^{x_1} = -\dfrac{3x_1}{2} - 2 \Rightarrow Be^{x_1} = -2$

$f'(x_1) = A + Be^{x_1} = m_1$

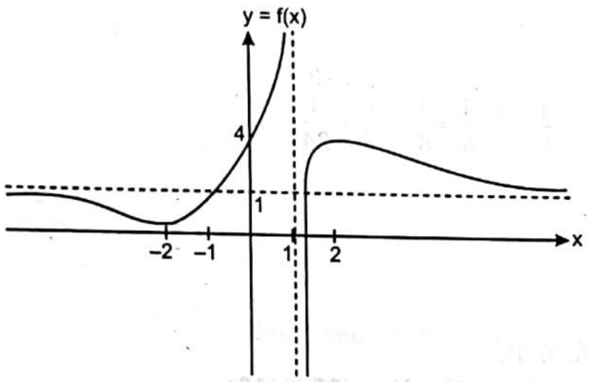
$$m_1 = -\frac{3}{2} - 2 = -\frac{7}{2}$$

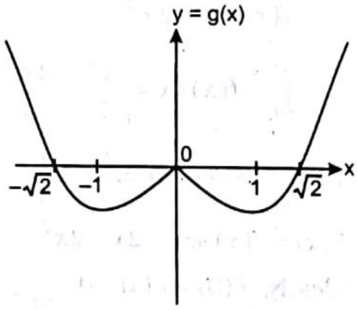
$$m_2 = -\frac{3}{2}$$

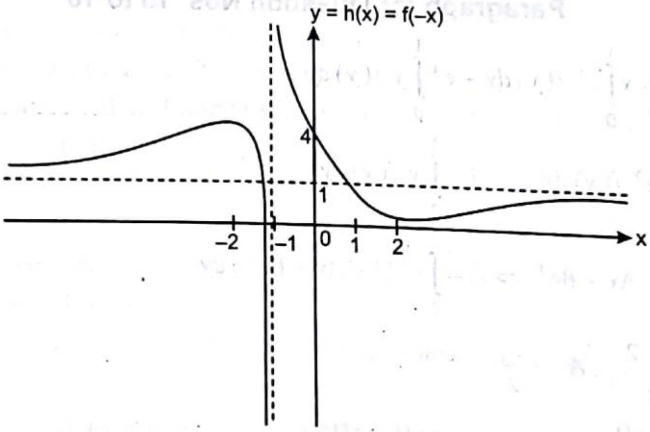
$$\tan\theta = \frac{8}{25}$$

Exercise-4 : Matching Type Problems

3.







4. (A) $y = \dfrac{x^3}{(x-\alpha)(x-\beta)(x-\gamma)}$

$$y = \frac{8}{(2-\alpha)(2-\beta)(2-\gamma)}$$

$$x^3 - 3x^2 + 2x + 4 = (x-\alpha)(x-\beta)(x-\gamma)$$

Put $x = 2$

$8 - 12 + 4 + 4 = (2-\alpha)(2-\beta)(2-\gamma) = 4$

$\therefore \quad y|_{\text{at } x=2} = \frac{8}{4} = 2$

(B) $x^3 + ax + 1 = 0$, $\quad x^4 + ax + 1 = 0 \quad \ldots(1)$

$x^4 + ax^2 + x = 0 \quad \ldots(2)$

(2) – (1) gives $\quad ax^2 - ax + x - 1 = 0$

$ax(x-1) + (x-1) = 0$

$(x-1)(ax+1) = 0$

$x = 1 \quad \text{or} \quad x = -\frac{1}{a}$

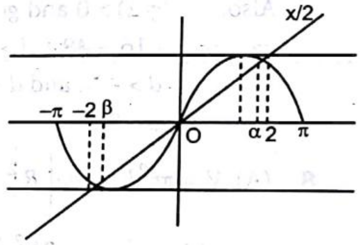
put $x = 1$ in (1) we get, $\quad 1 + a + 1 = 0 \Rightarrow a = -2$

$|a| = 2$

(C) $f(x) = x^2 + 4\cos x + 5$

$f'(x) = 2x - 4\sin x = 2(x - 2\sin x) = 0$

$\sin x = \frac{x}{2}$

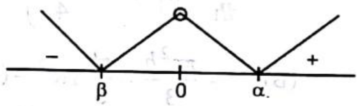


$f'(x)$ — (>0) when $x > \alpha$

— <0 when $0 < x < \alpha$

— >0 when $\beta < x < 0$

— <0 when $x < \beta$



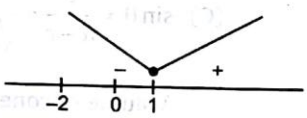
$x = 0$ only a point of maxima

So, number of local maxima is 1.

(D) Let $|x| = t \in [0, 2]$

$\therefore \quad f(x) = 2t^3 + 3t^2 - 12t + 1 = g(t)$

$g'(t) = 6t^2 + 6t - 12t = 6(t^2 + t - 2) = 6(t+2)(t-1)$



$g(1)$ is min. of $f(x)$ *i.e.*, $f_{\min} = 2 + 3 - 12 + 1 = -6$

$g(0) = 1$, $g(2) = 16 + 12 - 24 + 1 = 5$

max. of $f(x)$ is $5 > 4, 3, 2, 0$

6. $f' = \frac{1}{8x} - a + 2x > 0$ since $x > 0$

$1 - 8ax + 16x^2 > 0$

See $D=0$

$D<0$

$D>0$

7. Let $g(x)=ax^3+bx^2+cx+d$

$f(x)=\sqrt{g(x)}$

$f(x)$ has local maxima and local maxima at $x=-2$ and $x=2$.

$\Rightarrow$ $g(x)$ has same local minima and maxima and $x=-2$ and 2.

$\Rightarrow$ $a<0\,;a=-2$

$$f'(x)=\frac{3ax^2+2bx+c}{2\sqrt{ax^3+bx^2+cx+d}}=0$$

$f'(-2)=0$ and $f'(2)=0$

$\Rightarrow$ $b=0, c=24$

Also, $g(2)>0$ and $g(-2)>0$

$\Rightarrow$ $-16+48+d>0$ and $16-48+d>0$

$d>-32$ and $d>32$

$\Rightarrow$ $d>32$

8. (A) $V=\pi r^2h=\pi h\cdot\left(R^2-\frac{h^2}{4}\right)$ $\left(\because R^2=r^2+\frac{h^2}{4}\right)$

$$\frac{dV}{dh}=\pi\left(R^2-\frac{3h^2}{4}\right)=0\Rightarrow h=\frac{2R}{\sqrt{3}}\text{ and } r=\sqrt{\frac{2}{3}}R$$

(B) $V=\frac{\pi r^2h}{3}=\frac{\pi h}{3}\{R^2-(h-R)^2\}$ $\{\because R^2=r^2+(h-R)^2\}$

$$\frac{dV}{dh}=\frac{\pi}{3}(4hR-3h^2)=0\quad\Rightarrow h=\frac{4R}{3}\text{ and } r=\frac{2\sqrt{2}R}{3}$$

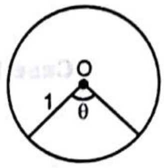
(C) $\sin\theta=\frac{r}{h-r}=\frac{R}{\sqrt{R^2+h^2}}\quad\Rightarrow R^2=\frac{h^2r^2}{h^2-2hr}$

$$\text{Volume of cone}=\frac{\pi}{3}R^2h=\frac{\pi}{3}\left(\frac{h^2r^2}{h-2r}\right)$$

$$\frac{dV}{dh}=\frac{\pi}{3}r^2\left(\frac{(h-2r)\,2h-h^2}{(h-2r)^2}\right)=0\quad\Rightarrow h=4r$$

(D) $$\frac{\left(\frac{2x}{3}+\frac{2x}{3}+\frac{2x}{3}\right)+\left(\frac{3y}{4}+\frac{3y}{4}+\frac{3y}{4}+\frac{3y}{4}\right)}{7}\ge\left(\frac{8x^3}{27}\cdot\frac{81y^4}{256}\right)^{1/7}\quad\Rightarrow x^3y^4\le\frac{32}{3}$$

Exercise-5 : Subjective Type Problems



1. $\dfrac{A_2}{A_1} = \dfrac{\pi r^2}{\left(\dfrac{2\pi - \theta}{2\pi}\right)\cdot \pi r^2} = \dfrac{2\pi}{2\pi - \theta}$

$V = \dfrac{\pi}{3}\left(\dfrac{\theta}{2\pi}\right)^2 \sqrt{1^2 - \left(\dfrac{\theta}{2\pi}\right)^2}$

$\dfrac{dV}{d\theta} = 0 \Rightarrow \theta = \sqrt{\dfrac{8}{3}}\,\pi$

$\dfrac{A_2}{A_1} = \dfrac{\sqrt{3}}{\sqrt{3} - \sqrt{2}} = 3 + \sqrt{6}$

2. $f(x) = x^2 \ln x$

$\Rightarrow \quad f'(x) = x(1 + 2\ln x)$

and $f'(x) > 0$ for $\in [1, e]$

$\therefore$ $f(x)$ is continuously increasing on $[1, e]$ with the least value zero at $x = 1$ and the greatest value e^2 at $x = e$.

3. $f(x) = px\,e^{-x} - \dfrac{x^2}{2} + x$

$f'(x) = (1 - x)[pe^{-x} + 1] \le 0$

$\Rightarrow \quad p \le -1$

4. $f'(x) = \begin{cases} ax\,e^{ax} + e^{ax}; & x \le 0 \\ 1 + 2ax - 3x^2; & x > 0 \end{cases}$

Clearly, $f'(x)$ is continuous at $x = 0$

$\Rightarrow \quad f''(x) = \begin{cases} a^2 x\,e^{ax} + 2ae^{ax}; & x \le 0 \\ 2a - 6x; & x > 0 \end{cases}$; $f'(x)$ increasing if $(ax + 2)\,ae^{ax} \ge 0$ and $2a - 6x \ge 0$

5. $f(x) = x^2 - 2bx + 1$

Case I : $b > 1$

$\Rightarrow \quad f(0) - f(1) = 4$

$\Rightarrow \quad 1 - (2 - 2b) = 4$

$\Rightarrow \quad b = \dfrac{5}{2}$

Case II : $0 < b < \dfrac{1}{2}$

$\Rightarrow \quad f(1) - f(b) = 4$

$\Rightarrow \quad b = 3, -1$ (Not possible)

Case III : $\frac{1}{2}<b<1$

$\Rightarrow \quad f(0)-f(b)=4$

$b=\pm 2 \quad$ (Not possible)

Case IV : $b<0$

$\Rightarrow \quad f(1)-f(0)=4$

$\Rightarrow \quad b=-\frac{3}{2}$

6. $x^2+9y^2=36 \qquad x^2+y^2=12$

$12-y^2+9y^2=36$

$8y^2=24 \Rightarrow y^2=3 \Rightarrow y=\sqrt{3},-\sqrt{3}$

when $y=\pm\sqrt{3}$, $x^2=12-3=9$

$x=\pm 3$

$\therefore$ point of intersections are $(\pm 3, \pm\sqrt{3})$

Let one of point of intersect is $(3, \sqrt{3})$

Now, $\frac{2x}{36}+\frac{2y}{4}y'=0 \Rightarrow y'=\left(-\frac{2x}{36}\right)\times\left(\frac{4}{2y}\right)$

$(y')_{(3,\sqrt{3})}=-\frac{1}{3\sqrt{3}}=(m_1)$

$2x+2yy'=0 \Rightarrow y'=-\frac{x}{y} \Rightarrow (y')_{(3,\sqrt{3})}=-\frac{3}{\sqrt{3}}=-\sqrt{3}=m_2$

$\tan\theta=\left|\frac{m_1-m_2}{1+m_2m_1}\right|=\left|\frac{-\frac{1}{3\sqrt{3}}+\sqrt{3}}{1+\frac{1}{3}}\right|=\left|\frac{8}{4\sqrt{3}}\right|=\left|\frac{2}{\sqrt{3}}\right|$

$\tan\theta=\frac{2}{\sqrt{3}} \Rightarrow \theta=\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

7. $f'(x)=2e^{2x}-(\lambda+1)e^x+2\geq 0 \ \forall\ x\in R$

i.e., $2e^{2x}+2-(\lambda+1)e^x$

$\lambda+1\leq 2(e^x+e^{-x})$

$\frac{\lambda+1}{2}\leq(e^x+e^{-x})\ \forall\ x\in R$

$\Rightarrow \quad \frac{\lambda+1}{2}\leq(e^x+e^{-x})_{\min}\ \forall\ x\in R$

So, $\frac{\lambda+1}{2} \le 2$

$\lambda + 1 \le 4$

$\lambda \le 3$

$\lambda \in (-\infty, 3]$

$\therefore \quad k = 3$

9. $f(x) = x^2, \qquad g(x) = -\frac{8}{x}$

$\Rightarrow q = p^2 \qquad \Rightarrow \quad s = -\frac{8}{r}$

Also, $\qquad \frac{s-q}{r-p} = 2p \quad \& \quad 2p = \frac{8}{r^2}$

$pr^2 = 4$ ...(1)

$\frac{s-q}{r-p} = 2p$

$\Rightarrow \quad \frac{-\frac{8}{r} - p^2}{r-p} = 2p \Rightarrow -\frac{8}{r} - p^2 = 2pr - 2p^2$

$p^2 = \frac{8}{r} + 2pr$

$\Rightarrow \qquad p^2 r = 16$ ...(2)

Multiply (1) & (2),

$\Rightarrow \qquad pr = 4$

$\Rightarrow \qquad r = 1, \quad p = 4$

10. $f(x) = \begin{bmatrix} |x+2|, & x \ge 0 \\ |x-2|, & x < 0 \end{bmatrix}$

Minimum value of $f(x)$ is 2.

11. $f(x) = \int_0^x [(a-1)(t^2+t+1)^2 - (a+1)(t^4+t^2+1)]\,dt$

$2\int_0^x (t^2+t+1)(at - t^2 - 1)\,dt$

$f'(x) = 2(x^2+x+1)(ax - x^2 - 1) = 0$

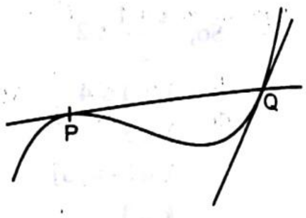
$D < 0 \Rightarrow a^2 - 4 < 0$

12. $f(x) = x^{2013} + e^{2014x}$

$f'(x) = 2013\,x^{2012} + 2014\,e^{2014x} > 0$

$\Rightarrow \quad f(x)$ is increasing function.

14. $P=(x_1, x_1^3 - ax_1)$



$Q=(x_2, x_2^3 - ax_2)$

$y = x^3 - ax$

$\frac{dy}{dx} = 3x^2 - a$

Slope at P = slope of PQ

$$\therefore \quad (3x_1^2 - a) = \left(\frac{x_2^3 - ax_2 - x_1^3 + ax_1}{x_2 - x_1}\right) \quad (\because x_1 \neq x_2)$$

$$(x_2 - x_1)(x_2 + 2x_1) = 0$$

$$\Rightarrow \quad x_2 = -2x_1 \quad \text{...(1)}$$

Slope at $P \times$ Slope at $Q = -1$

$$(3x_1^2 - a)(3x_2^2 - a) = -1 \quad \text{...(2)}$$

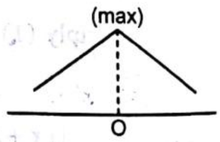
Put (1) in (2),

$$36x_1^4 - 15ax_1^2 + (a^2 + 1) = 0$$

$$D \geq 0 \quad (\because x_1 \in R)$$

$$9a^2 \geq 16 \Rightarrow a \geq \frac{4}{3}$$

15. $I(t) = \int_{\alpha}^{\beta} (x^2 + 2x - t^2)\, dx = \frac{x^3}{3} + x^2 - t^2 x \Big|_{\alpha}^{\beta}$



$I(t) = \frac{\beta^3 - \alpha^3}{3} + (\beta^2 - \alpha^2) - t^2(\beta - \alpha)$

$I'(t) = -2t(\beta - \alpha) = 0$

$I(t) \leq I(0)$

$\therefore \quad I(0) = \int_{-2}^{0} (x^2 + 2x)\, dx = -\frac{4}{3} \Rightarrow \frac{p}{q} = |I(0)| = \frac{4}{3}$

16. $\frac{dy}{dt} = 1 - \frac{3y}{100 - 2t}$ when $t = 0, y = 0$

$\frac{dy}{dt} + \left(\frac{3}{100 - 2t}\right) y = 1$

$y(100 - 2t)^{-3/2} = +(100 - 2t)^{-1/2} + c$

as when $t = 0, y = 0$

$c = -\frac{1}{10}$

$y = (100 - 2t) - \frac{1}{10}(100 - 2t)^{3/2}$

$$\frac{dy}{dt} = -2 - \frac{1}{10}\left(\frac{3}{2}\right)(100-2t)^{1/2}(-2) = 0$$

$$t = \frac{250}{9} = 27 + \frac{7}{9}$$

17. Let $f'(x) = K$

$\Rightarrow \quad f(x) = Kx + c$

$\Rightarrow \quad f(9) - f(-3) = 12K$

Maximum value of $f(9) - f(-3) = 96$

19. Equation of normal at $P\left(\frac{3}{4}y_1^3, y_1\right)$ is $y - y_1 = \frac{-9y_1^2}{4}\left(x - \frac{3}{4}y_1^3\right)$

If it passes from (0, 1) then $27y_1^5 + 16y_1 - 16 = 0$ has only one real root.

20. $e^{-x}\left(\frac{x^2}{2} + x + 1\right) = a$

Let $f(x) = e^{-x}\left(\frac{x^2}{2} + x + 1\right)$

$$f'(x) = e^{-x}\left(-\frac{x^2}{2}\right) < 0$$

21. $f'(x) = a - 2\sin 2x + \cos x - \sin x$

Let $g(x) = -2\sin 2x + \cos x - \sin x$

$= -2\{(\cos x - \sin x)^2 - 1\} + \cos x - \sin x$

where $\cos x - \sin x = t$

$-2t^2 + t + 2 \ \forall\, t \in [-\sqrt{2}, \sqrt{2}]$

$-2 - \sqrt{2} \le g(x) \le \frac{17}{8} \Rightarrow a \ge \frac{17}{8}$

22. Let $x = 6\cos^3\theta, \quad y = 6\sin^3\theta$

$$\frac{dy}{dx} = \frac{6(3\sin^2\theta\cos\theta)}{-6(3\cos^2\theta\sin\theta)} = -\tan\theta$$

Equ. of tangent

$y - 6\sin^3\theta = -\tan\theta(x - 6\cos^3\theta) \Rightarrow p_1 = 6\sin\theta\cos\theta$

Equ. of normal

$y - 6\sin^3\theta = \cot\theta(x - 6\cos^3\theta) \quad \Rightarrow p_2 = 6(\cos^2\theta - \sin^2\theta)$

$$\sqrt{4p_1^2 + p_2^2} = 6\sqrt{4\sin^2\theta\cos^2\theta + \cos^4\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta} = 6$$

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Chapter 5 – Indefinite and Definite Integration

5 INDEFINITE AND DEFINITE INTEGRATION

Exercise-1 : Single Choice Problems

1. $\int\left[a^x \ln x + \underbrace{a^x \ln a}_{\text{II}} \cdot \underbrace{x(\ln x - 1)}_{\text{I}}\right] dx$

$$= \int a^x \cdot \ln x\, dx + \left[x(\ln x - 1)\, a^x - \int\left[x \cdot \frac{1}{x} + (\ln x - 1)\right] a^x\right] dx$$

$$= \int a^x \cdot \ln x\, dx + [x(\ln x/e)]\, a^x - \int (\ln x)\, a^x dx$$

2. $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} \frac{1}{\sqrt{1+\frac{r}{n}}} = \int\limits_0^1 \frac{dx}{\sqrt{x+1}} = 2(\sqrt{2}-1)$

3. $\int \frac{\sin x}{\sin(x-\alpha)} dx$

Let $x - \alpha = t \Rightarrow dx = dt$

$$\int \frac{\sin(t+\alpha)}{\sin t} dt = t\cos\alpha + \sin\alpha \log\sin t + C = x\cos\alpha + \sin\alpha\log\sin(x-\alpha) + C$$

4. $\int\limits_0^2 \frac{\log(x^2+2)}{(x+2)^2} dx = \left(\frac{-\log(x^2+2)}{x+2}\right)_0^2 + \int\limits_0^2 \frac{2x\,dx}{(x+2)(x^2+2)}$

$$= \frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2} - \frac{5}{12}\log 2 + \frac{1}{12}\log 3$$

5. For $0 < x < 1$

$$1 + x^9 < 1 + x^8 < 1 + x^4 < 1 + x^3$$

6. Let $g(x) = \int\limits_0^x \sqrt{1-(f(s))^2}\, ds$

$$\lim_{t\to x}\left(\frac{g(t)-g(x)}{f(t)-f(x)}\right)=f(x) \quad \Rightarrow g'(x)=f(x)\,f'(x)=\sqrt{1-(f(x))^2}$$

$$\int\frac{y\,dy}{\sqrt{1-y^2}}=\int dx \quad \Rightarrow \sqrt{1-y^2}=1-x \qquad \left(\because f\left(\frac{1}{2}\right)=\frac{\sqrt{3}}{2}\right)$$

7. $$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{f\left(\frac{r}{n}\right)}=\int_0^1\sqrt{f(x)}\,dx=\frac{2}{\sqrt{3}}\int_0^1\sqrt{1-\cos^6 x-\sin^6 x}\,dx$$

$$=\int_0^1\sin 2x\,dx=\frac{1-\cos 2}{2}$$

8. $$\int_0^1\frac{(x^6-x^3)}{(2x^3+1)^3}dx=\frac{1}{2}\int_0^1\frac{2\left(1-\frac{1}{x^3}\right)}{\left(2x+\frac{1}{x^2}\right)^3}dx=-\frac{1}{36} \qquad \left(\text{Put } 2x+\frac{1}{x^2}=t\right)$$

9. $$2\int_0^{1/\sqrt{2}}\frac{\sin^{-1}x}{x}dx-\int_0^1\frac{\tan^{-1}x}{x}dx$$

$$2\int_0^{\pi/4}\frac{\theta\cos\theta}{\sin\theta}d\theta-\int_0^{\pi/4}\frac{\theta\sec^2\theta}{\tan\theta}d\theta=-\frac{\pi}{4}\ln 2-2\int_0^{\pi/4}\ln\sin\theta\,d\theta+\int_0^{\pi/4}\ln\tan\theta\,d\theta$$

$$=-\int_0^{\pi/4}\ln\sin 2\theta\,d\theta=\frac{\pi}{4}\ln 2$$

10. $$f(x)=x^2+\int_0^x e^{-t}f(x-t)\,dt=x^2-e^{-x}\int_x^0 e^u f(u)\,du \qquad (\text{Let } x-t=u)$$

$$f'(x)=2x+e^{-x}\int_x^0 e^u f(u)\,du+f(x) \quad \Rightarrow f'(x)=x^2+2x$$

$$\Rightarrow f(x)=\frac{x^3}{3}+x^2$$

11. $$f'(x)=f(x)+k_1 \qquad \left(k_1=\int_0^1 f(x)\,dx\right)$$

$$\Rightarrow y=ke^x-k_1$$

If $f(0)=1 \Rightarrow k-k_1=1$

$$k_1=\int_0^1(ke^x-k_1)\,dx \Rightarrow 2k_1=k(e-1) \Rightarrow k=\frac{2}{3-e} \text{ and } k_1=\frac{e-1}{3-e}$$

12. $I_1 = \int_{\sin^2 x}^{1+\cos^2 x} tf(t(2-t))\,dt = 2\int_{\sin^2 x}^{1+\cos^2 x} f(t(2-t))\,dt - \int_{\sin^2 x}^{1+\cos^2 x} tf(t(2-t))\,dt$

$I_1 = 2I_2 - I_1 \Rightarrow I_1 = I_2$

13. $\int \frac{5\sin x\,dx}{\sin x - 2\cos x} = \int dx + 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x}dx = x + 2\ln|\sin x - 2\cos x| + C$

14. $\int \frac{(2+\sqrt{x})\,dx}{(x+1+\sqrt{x})^2} = \int \frac{\frac{2}{x^2} + \frac{1}{x^{3/2}}}{\left(\frac{1}{x} + \frac{1}{\sqrt{x}} + 1\right)^2}dx$

Let $\frac{1}{x} + \frac{1}{\sqrt{x}} + 1 = t \Rightarrow \left(-\frac{1}{x^2} - \frac{1}{2x^{3/2}}\right)dx = dt$

15. $\int \frac{\left(\sqrt[3]{x+\sqrt{2-x^2}}\right)\left(\sqrt[6]{1-x\sqrt{2-x^2}}\right)dx}{\sqrt[3]{1-x^2}} = \int \frac{\sqrt[3]{x+\sqrt{(2-x^2)}}\sqrt[3]{\left(\frac{\sqrt{2-x^2}-x}{2}\right)}}{\sqrt[3]{1-x^2}}dx$

$= 2^{1/6}\int dx = 2^{1/6}x + C$

16. $\int \frac{dx}{\sqrt{1-\tan^2 x}}$

$\int \frac{\cos x\,dx}{\sqrt{1-2\sin^2 x}} = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}\sin x) + C$

17. $I = \int \frac{dx}{x^{5/6}\cdot(x+1)^{7/6}}$

$\frac{x}{x+1} = t \qquad \frac{dx}{(x+1)^2} = dt$

$I = \int \frac{(x+1)^2\,dt}{[t(x+1)]^{5/6}(x+1)^{7/6}} = \int t^{-5/6}dt = 6t^{1/6} + C$

18. $I_n = \int \sin^n x\,dx = \int \sin^{n-2} x(1-\cos^2 x)\,dx$

$I_n = I_{n-2} - \int \underbrace{\sin^{n-2} x\cdot\cos x}_{\text{II}} \cdot \underbrace{\cos x}_{\text{I}}\,dx$

$I_n = I_{n-2} - \left(\frac{\cos x\cdot\sin^{n-1} x}{n-1} - \int -\sin x\cdot\frac{\sin^{n-1} x}{n-1}dx\right)$

$I_n = I_{n-2} - \frac{\cos x\cdot\sin^{n-1} x}{n-1} - \frac{1}{n-1}I_n$

$nI_n - (n-1)I_{n-2} = -\cos x\cdot\sin^{n-1} x$

19. $\int x^2 \frac{1}{(a+bx)^2}dx$ Let $a+bx=t$ then $dx=\frac{dt}{b}$

$$\therefore \int x^2 \frac{1}{(a+bx)^2}dx=\int\left(\frac{t-a}{b}\right)^2\cdot\frac{1}{t^2}\cdot\frac{dt}{b}=\frac{1}{b^3}\int\left(\frac{t^2-2at+a^2}{t^2}\right)dt$$

$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt=\frac{1}{b^3}\left[t-2a\ln|t|-\frac{a^2}{t}\right]+C$$

$$=\frac{1}{b^3}\left[a+bx-2a\ln|a+bx|-\frac{a^2}{a+bx}\right]+C$$

20. $\int \frac{8x^{43}+13x^{38}}{(x^{13}+x^5+1)^4}dx$

$\int \frac{8x^{-9}+13x^{-14}}{(1+x^{-8}+x^{-13})^4}dx$

Let $1+x^{-8}+x^{-13}=t$

$(-8x^{-9}-13x^{-14})\,dx=dt$

$$\therefore \int -\frac{dt}{t^4}=+\frac{1}{3t^3}+C=\frac{1}{3(1+x^{-8}+x^{-13})^3}+C=\frac{x^{39}}{3(x^{13}+x^5+1)^{13}}+C$$

21. $\int\left(\frac{(\cos 6x+\cos 4x)+5(\cos 4x+\cos 2x)+10(\cos 2x+1)}{10\cos^2 x+5\cos x\cos 3x+\cos x\cos 5x}\right)dx$

$$=\int \frac{2\cos 5x\cos x+5(2\cos 3x\cos x)+10(2\cos^2 x)}{10\cos^2 x+5\cos x\cos 3x+\cos x\cos 5x}dx$$

$$=\int 2\,dx=2x+C$$

$f(x)=2x \Rightarrow f(10)=20$

22. $\int (1+x-x^{-1})e^{x+x^{-1}}dx=\int e^{x+x^{-1}}\cdot 1\,dx+\int (x+x^{-1})e^{x+x^{-1}}dx$

$$=e^{x+x^{-1}}\cdot x-\int e^{x+x^{-1}}\left(1-\frac{1}{x^2}\right)x\,dx+\int (x-x^{-1})e^{x+x^{-1}}dx+C$$

$$=xe^{x+x^{-1}}+C$$

23. $\int e^x\left[\frac{2\tan x}{1+\tan x}+\frac{1}{\left(\frac{1-\cos(\pi/2+2x)}{2}\right)}\right]dx=\int e^x\left[\frac{2\tan x}{1+\tan x}+\frac{2}{(1+\sin 2x)}\right]dx$

$$=2\int e^x\left[\frac{\sin x}{\sin x+\cos x}+\frac{1}{(\sin x+\cos x)^2}\right]dt$$

Let $f(x) = \dfrac{\sin x}{\sin x + \cos x}$, $f'(x) = \dfrac{1}{(\sin x + \cos x)^2}$

$$= 2 \cdot e^x \left(\frac{\sin x}{\sin x + \cos x} \right) + C$$

$$g\left(\frac{5\pi}{4}\right) = 1$$

24. $\dfrac{d}{dx}(x \sin x + \cos x) = x \cos x \qquad f'(x) = x \cos x$

Let $f(x) = x \sin x + \cos x \qquad f''(x) = -x \sin x + \cos x$

$$\int e^{f(x)} \left(xf'(x) + \frac{f''(x)}{(f'(x)^2)} \right) dx = \int xe^{f(x)} f'(x)\, dx + \int e^{f(x)} \cdot \frac{f''(x)}{(f'(x)^2)} dx$$

$$= xe^{f(x)} - \int e^{f(x)} dx + e^{f(x)} \frac{-1}{f'(x)} - \int e^{f(x)} \cdot f'(x) \left(\frac{-1}{f'(x)} \right) dx$$

$$= xe^{f(x)} - \frac{e^{f(x)}}{f'(x)} + C = e^{f(x)} \left(x - \frac{1}{f'(x)} \right) + C$$

$$= e^{x \sin x + \cos x} \left(x - \frac{1}{x \cos x} \right) + C$$

25. $\displaystyle\int_0^1 \left(\sqrt{x} + \frac{1}{\sqrt{x} + \sqrt{1+x}} \right) dx$

$$\int_0^1 (\sqrt{x} + (\sqrt{1+x} - \sqrt{x})\, dx = \int_0^1 (\sqrt{1+x}\, dx = \frac{2}{3}(1+x)^{3/2} \Big|_0^1 = \frac{2}{3}(2^{3/2} - 1)$$

26. $\displaystyle\int x^{x^2} x(2 \ln x + 1)\, dx$

$x^{x^2} = t$

$x^2 \ln x = \ln t$

$\left(x^2 \dfrac{1}{x} + (\ln x) 2x \right) dx = \dfrac{1}{t} dt$

$\therefore \quad \displaystyle\int t \cdot \frac{dt}{t} = \int dt = t + C = x^{x^2} + C = (x^x)^x + C$

27. $= \displaystyle\int \sec^{2010} x \operatorname{cosec}^2 x\, dx - \int 2010 \sec^{2010} x\, dx$

$$= \int \sec^{2010} x(-\cot x) - \int 2010 \sec^{2010} x \cdot \tan x \cdot (-\cot x) - \int 2010 \sec^{2010} x\, dx$$

$$= -\frac{\cot x}{(\cos x)^{2010}} + 2010 \int \sec^{2010} x\, dx - 2010 \int \sec^{2010} x\, dx + C = \frac{-\cot x}{(\cos x)^{2010}} + C$$

$\therefore \quad \dfrac{f(x)}{g(x)} = \dfrac{1}{\sin x} = \{x\}$ no solution.

28. Let $x^x \ln x = t$

$\Rightarrow \left(x^x \ln x(1+\ln x) + \dfrac{x^x}{x}\right)dx = dt$

$\Rightarrow x^x\left(\ln x + (\ln x)^2 + \dfrac{1}{x}\right)dx = dt$

$\int dt = t + C = x^x \ln x + C$

29. $I = \int \dfrac{\left(\dfrac{1}{x^3} - \dfrac{1}{x^5}\right)dx}{\sqrt{2 - \dfrac{2}{x^2} + \dfrac{1}{x^4}}}$

Let $2 - \dfrac{2}{x^2} + \dfrac{1}{x^4} = t \Rightarrow I = \dfrac{1}{4}\int \dfrac{dt}{\sqrt{t}}$

30. Put $\ln x = t$

$I = \int e^t\left(\dfrac{t-1}{t^2+1}\right)^2 dt = \int e^t\left(\dfrac{1}{t^2+1} - \dfrac{2t}{(t^2+1)^2}\right)dt$

31. $I = \int \dfrac{dx}{(x-1)^{3/4}(x+2)^{5/4}} = \int \dfrac{dx}{\left(\dfrac{x-1}{x+2}\right)^{3/4}(x+2)^2}$

Let $\dfrac{x-1}{x+2} = t \Rightarrow dt = \dfrac{3dx}{(x+2)^2}$

32. $\int \dfrac{2-(1+x^7)}{x(1+x^7)}dx = -\int \dfrac{dx}{x} + \dfrac{2}{7}\int \dfrac{7x^6}{x^7(1+x^7)}dx = -\ln|x| + \dfrac{2}{7}\ln|1+x^7| + C$

33. $I = \int \dfrac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x\cos^2 x}dx = \int (\sin^2 x - \cos^2 x)\,dx = -\int \cos 2x\,dx$

34. $I = 2^{1/3}\int \dfrac{(\tan x)^{1/3}\,d((\tan x)^{1/3})}{(\tan x)^{2/3}+1}$

Let $(\tan x)^{1/3} = t \Rightarrow d((\tan x)^{1/3}) = dt$

$I = \dfrac{2^{1/3}}{2}\int \dfrac{2t}{t^2+1}dt$

35. $\int \dfrac{(2012)^x}{\sqrt{1-(2012)^{2x}}}\cdot(2012)^{\sin^{-1}(2012)^x}dx$

Let $\sin^{-1}(2012)^x = t \Rightarrow \dfrac{1}{\ln 2012}\int (2012)^t\,dt = \dfrac{(2012)^{\sin^{-1}(2012)^x}}{\ln^2(2012)} + C$

36. Let $x+1=t^2 \Rightarrow dx=2t\,dt$

$$2\int\frac{(t^2+1)\,dt}{t^4+t^2+1}=2\int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+3}$$

37. $\int\left(\frac{f(x)g'(x)-f'(x)g(x)}{f(x)g(x)}\right)\ln\left(\frac{g(x)}{f(x)}\right)dx$

Let $\frac{g(x)}{f(x)}=t \Rightarrow \frac{f(x)g'(x)-g(x)f'(x)}{(f(x))^2}dx=dt;\quad \int\frac{\ln t}{t}dt=\frac{(\ln t)^2}{2}+C$

38. $\int\left(\int e^x\left(\ln x+\frac{2}{x}-\frac{1}{x^2}\right)dx\right)dx$

$$\int\left(\int e^x\left(\ln x+\frac{1}{x}\right)dx+\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx\right)dx=\int\left(e^x\left(\ln x+\frac{1}{x}\right)+C_1\right)dx=e^x\ln x+C_1x+C_2$$

39. $f(x)=\pi^2\left(\left|\frac{-t\cos(x+\pi t)}{\pi}\right|_0^1+\int_0^1\frac{1\cdot\cos(x+\pi t)}{\pi}dt\right)=\pi\cos x-2\sin x$

40. $\frac{2}{x}\le\sqrt{5}\Rightarrow x\in\left[\frac{2}{\sqrt{5}},1\right)$

$$\therefore\ \int_0^1 f(x)\,dx=\int_0^{2/\sqrt{5}}f(x)\,dx+\int_{2/\sqrt{5}}^1 f(x)\,dx\le\sqrt{5}\left(\frac{2}{\sqrt{5}}-0\right)+\int_{2/\sqrt{5}}^1\frac{2}{x}dx$$

$$\therefore\ \int_0^1 f(x)\,dx\le 2+2\,[\ln x]_{2/\sqrt{5}}^1$$

$$\therefore\ a=2+2\ln\left[\frac{\sqrt{5}}{2}\right]$$

42. $f(0)=0,\ f(2\pi)=2\pi$

$$\therefore\ \int_0^{2\pi}f(x)\,dx+\int_0^{2\pi}f^{-1}(x)\,dx=\int_0^{2\pi}2\pi\,dx=4\pi^2$$

$$\Rightarrow\ \left[\frac{x^2}{2}-\cos x\right]_0^{2\pi}+I=4\pi^2\ \Rightarrow\ I=\int_0^{2\pi}f^{-1}(x)\,dx=2\pi^2$$

43. $=2\left[2\int_0^1 e^{-x^4}dx-\int_0^1 8x^4e^{-x^4}dx\right]=2\left[2\left[(xe^{-x^4})_0^1+\int_0^1 4x^4e^{-x^4}dx\right]-\int_0^1 8x^4e^{-x^4}dx\right]$

$=\frac{4}{e}$

46. Put $y-2=z$

$$I=\int_{-2}^{2}\frac{z^2+1}{2z^2+3}\sin(z)\,dz=0$$

47. $\int_1^4 \frac{3}{x}e^{\sin x^3}dx$

Let $x^3=t \Rightarrow 3x^2dx=dt$

$$\int_1^{64}\frac{e^{\sin t}}{t}dt=F(64)-F(1)$$

51. $\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}dt=\lim_{x\to\infty}\frac{x\cdot\int_0^x e^{t^2}dt}{e^{x^2}}$

Apply L' Hospital's rule,

$$\lim_{x\to\infty}\frac{x\cdot(e^{x^2})+\int_0^x e^{t^2}dt\cdot 1}{e^{x^2}\cdot 2x}=\lim_{x\to\infty}\left(\frac{1}{2}+\frac{\int_0^x e^{t^2}dt}{2xe^{x^2}}\right)=\frac{1}{2}$$

52. $L=\sum_{r=1}^{n}\frac{2\cdot r+n}{r^2+n\cdot r+n^2}=\int_0^1\frac{(2x+1)\,dx}{x^2+x+1}=\ln(x^2+x+1)|_0^1$

$L=\ln 3$

53. Let $\sqrt[3]{x^2+2x}=y=f(x)$

$x=-1+(y^3+1)^{1/2}$

$$I=\int_0^2(f^{-1}(x)+f(x)+1)\,dx$$

Consider $\int_0^2 f^{-1}(x)=\int_0^2 tf'(t)\,dt$ Let $f^{-1}(x)=t; x=f(t); dx=f'(t)\,dt$

$$=tf(t)|_0^2-\int_0^2 dx=6$$

54. Put $x=2\tan\theta$ then $I=\int_0^{\pi/2}\left(\frac{\ln 2+\ln\tan\theta}{4\sec^2\theta}\right)2\sec^2\theta\,d\theta$ then solve it.

55. Put $x-5=t$

$x=0, t=-5$

$x=10, t=5$

$$\int_{-5}^{5}(t+t^2+t^3)\,dt=\left.\frac{t^3}{3}\right|_{-5}^{5}=\frac{250}{3}$$

56. Let $$I=\int_0^\infty \frac{dx}{(1+x^9)(1+x^2)} \quad \text{...(1)}$$

Put $x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt$

$$I=\int_\infty^0 \frac{-\frac{dt}{t^2}}{\left(\frac{t^9+1}{t^9}\right)\left(\frac{1+t^2}{t^2}\right)}=\int_0^\infty \frac{t^9\,dt}{(t^9+1)(1+t^2)} \quad \text{...(2)}$$

On adding (1) & (2),

$$2I=\int_0^\infty \frac{dt}{(1+t^2)}=\left.\tan^{-1}t\right|_0^\infty$$

$$2I=\frac{\pi}{2} \Rightarrow I=\frac{\pi}{4}$$

57. $$I=\int_0^{\pi/2}\left(\frac{1+\sin 3x}{1+2\sin x}\right)dx=\int_0^{\pi/2}\frac{1+3\sin x-4\sin^3 x}{1+2\sin x}dx$$

$$=\int_0^{\pi/2}\frac{(1+2\sin x)(-2\sin^2 x+\sin x+1)}{(1+2\sin x)}=-2\left(\frac{1}{2}\frac{\pi}{2}\right)+1+\frac{\pi}{2}=1$$

58. $$\lim_{x\to\infty}\frac{(\tan^{-1}x)^2}{\frac{1}{2\sqrt{x^2+1}}2x}$$

$$\lim_{x\to\infty}(\tan^{-1}x)^2\frac{\sqrt{1+x^2}}{x}=\frac{\pi^2}{4}$$

59. Let $t=\prod_{r=1}^{2013}(x^2+r^2)$

$$\ln t=\sum_{r=1}^{2013}\ln(x^2+r^2)$$

$$\frac{1}{t}dt=\sum_{r=1}^{2013}\frac{2x}{x^2+r^2}dx \Rightarrow dt=2\left(\sum_{r=1}^{2013}\frac{x}{x^2+r^2}\right)t\,dx$$

$$\therefore \quad \int \frac{dt}{2} = \frac{t}{2} = \frac{1}{2}\left(\prod_{r=1}^{2013}(x^2+r^2)\right)_0^1 = \frac{1}{2}\left(\prod_{r=1}^{2013}(1+r^2) - \prod_{r=1}^{2013} r^2\right)$$

60. $f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$

$$= \frac{2(1+x^2)-1-\sqrt{1+x^2}}{1+x^2} = \frac{\left((1+x^2)-\sqrt{1+x^2}\right)+x^2}{1+x^2} > 0 \ \forall \ x \in R$$

61. $I = \int_0^{\pi/2} \frac{\sin x \cos x \, dx}{1+\sin x + \cos x} = \int_0^{\pi/2} \sin\frac{x}{2}\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) dx$

$$= \frac{1}{2}\int_0^{\pi/2} [\sin x + \cos x - 1]\, dx = 1 - \frac{\pi}{4}$$

62. $I = \int_3^7 \frac{\cos x^2 dx}{\cos x^2 + \cos(10-x)^2}$

$$I = \int_3^7 \frac{\cos(10-x)^2 dx}{\cos(10-x)^2 + \cos x^2}$$

$\Rightarrow \quad 2I = 4 \quad \Rightarrow I = 2$

63. $-\int_{e^{-1}}^{1} \frac{\ln x}{x} dx + \int_1^{e^2} \frac{\ln x}{x} dx = -\left[\frac{(\ln x)^2}{2}\right]_{e^{-1}}^{1} + \left[\frac{(\ln x)^2}{2}\right]_1^{e^2} = 2 + \frac{1}{2} = \frac{5}{2}$

64. $\lim_{x\to\frac{\pi}{4}} \frac{\int_2^{\operatorname{cosec}^2 x} tg(t)\, dt}{x^2 - \frac{\pi^2}{16}} = \lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x\, g(\operatorname{cosec}^2 x)\, 2\operatorname{cosec} x(-\operatorname{cosec} x)\cot x}{2x} = \frac{-16}{\pi} g(2)$

65. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n-k}{n^2} \cos\frac{4k}{n}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\left(1-\frac{k}{n}\right)}{n} \cos\left(\frac{4k}{n}\right) = \int_0^1 (1-x)\cos 4x\, dx$$

$$= (1-x)\frac{\sin 4x}{4}\Big|_0^1 + \int_0^1 \frac{\sin 4x}{4} dx = -\frac{1}{16}\cos 4x\Big|_0^1$$

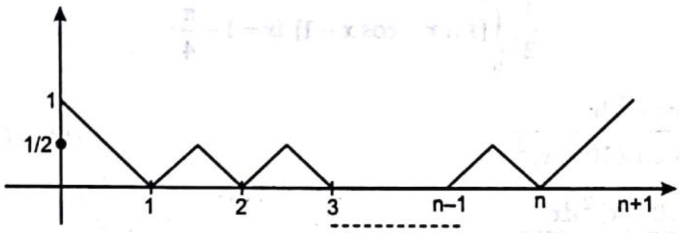
$$= -\frac{1}{16}(\cos 4 - 1) = \frac{1}{16}(1-\cos 4)$$

66. $\lim_{n\to\infty}\left[\int_0^{1/n}\sin\frac{\pi}{2n}dx+\int_{1/n}^{2/n}\sin\frac{2\pi}{2n}dx+\ldots\ldots+\int_{1-1/n}^{1}\sin\frac{n\pi}{2n}dx\right]$

$\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots\ldots+\sin\frac{n\pi}{2n}\right]$

$\lim_{n\to\infty}\frac{\sin\left(\frac{n\pi}{4n}\right)}{n\sin\frac{\pi}{4n}}\sin\left(\frac{(n+1)\pi}{4n}\right)=\frac{2}{\pi}$

67. $\int_0^{n+1}\min.\{|x-1|,|x-2|,|x-3|,\ldots\ldots|x-n|\}dx=\frac{1}{2}(1)+\frac{1}{2}\times\frac{1}{2}\times(n-1)+\frac{1}{2}\times(1)=\frac{n+3}{4}$



68. $S_k=\frac{1}{2}k\sin\left(\frac{k\pi}{2n}\right)$

$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{2}k\sin\frac{k\pi}{2n}=\int_0^1\frac{1}{2}x\sin\left(\frac{\pi x}{2}\right)dx=\frac{2}{\pi^2}$

71. $f(x)=\int_0^{g(x)}\frac{dt}{\sqrt{1+t^3}}$ $\qquad g'(x)=-\sin x\cdot(1+\sin(\cos x))^2$

$f'(x)=g'(x)\cdot\frac{1}{\sqrt{1+(g(x))^3}}$

$f'\left(\frac{\pi}{2}\right)=g'\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$

72. $x^2f(x)=\int_4^x(4t^2-2f'(t))\,dt$

$x^2f'(x)+2x\,f(x)=4x^2-2f'(x)$

$16f'(4)+8f(4)=64-2f'(4)$

$18f'(4)=64$

$9f'(4)=32$

73. $\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{r^2}{n^3+r^3}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}=\int_0^2\frac{x^2dx}{1+x^3}=\left|\frac{1}{3}\ln|1+x^3|\right|_0^2=\frac{1}{3}\ln 9$

74. $\int_0^{2\pi}\cos^{-1}(\cos x)\,dx=2\int_0^{\pi}\cos^{-1}(\cos x)\,dx=2\cdot\left|\frac{x^2}{2}\right|_0^{\pi}=\pi^2$

75. $2f(x)=\int_0^x(x^2-2xt+t^2)\,g(t)\,dt$

$2f(x)=x^2\int_0^x g(t)\,dt-2x\int_0^x t\cdot g(t)\,dt+\int_0^x t^2g(t)\,dt$

$2f'(x)=x^2\cdot g(x)+\int_0^x g(t)\,dt\cdot 2x-2x(x\,g(x))-\int_0^x t\cdot g(t)\,dt\cdot 2+x^2g(x)$

$2f'(x)=2x\int_0^x g(t)\,dt-2\int_0^x tg(t)\,dt$

$f''(x)=x\cdot g(x)+\int_0^x g(t)\,dt-xg(x)$

$f''(x)=\int_0^x g(t)\,dt$

$f'''(x)=g(x)$

76. $I=\int_0^{\pi}\frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx=\lambda\int_0^{\pi/2}\sin^2 x\,dx$

$I=\int_0^{\pi}\frac{(\pi-x)^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx\ \Rightarrow\ 2I=\int_0^{\pi}\pi\cos^4 x\sin^2 x\,dx$

77. $\frac{1}{2}\cdot\left|\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right|_0^{\sqrt{3}}=\left|\tan^{-1}x\right|_0^{\sqrt{3}}=\frac{\pi}{3}-0=\frac{\pi}{3}$

78. $\int_0^3\{x\}^{[x]}dx=\int_0^3(x-[x])^{[x]}dx=\int_0^1 1\cdot dx+\int_1^2(x-1)\,dx+\int_2^3(x-2)^2dx$

79. $I=\int_0^1\frac{\tan^{-1}x}{x}dx$

$x=\tan\theta$

$$I=\int_0^{\pi/4}\frac{\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta=\int_0^{\pi/4}\frac{2\theta}{\sin 2\theta}d\theta=\frac{1}{2}\int_0^{\pi/2}\frac{t}{\sin t}dt$$

80. $$\int_0^{4/\pi}\underbrace{3x^2}_{\text{II}}\underbrace{\sin\frac{1}{x}}_{\text{I}}dx-\int_0^{4/\pi}x\cos\frac{1}{x}dx=\left|\sin\frac{1}{x}\cdot x^3\right|_0^{4/\pi}-\int_0^{4/\pi}\cos\frac{1}{x}\cdot\left(\frac{-1}{x^2}\right)\cdot x^3dx-\int_0^{4/\pi}x\cos\frac{1}{x}dx$$

$$\frac{64}{\pi^3}\cdot\frac{1}{\sqrt{2}}-\lim_{x\to 0}\left(x^3\sin\frac{1}{x}\right)=\frac{32\sqrt{2}}{\pi^3}$$

81. $$\int_{-1}^{x}\left(8t^2+\frac{28t}{3}+4\right)dt=\frac{\frac{3x}{2}+1}{\log_{(x+1)}\sqrt{x+1}}$$

$$\left|\frac{8t^3}{3}+\frac{14t^2}{3}+4t\right|_{-1}^{x}=\frac{\frac{3x}{2}+1}{\frac{1}{2}}$$

$$\frac{8x^3}{3}+\frac{14x^2}{3}+4x-\left(\frac{-8}{3}+\frac{14}{3}-4\right)=3x+2$$

$$8x^3+14x^2+12x+8-14+12=9x+6$$

$$8x^3+14x^2+3x=0$$

$$x(8x^2+14x+3)=0$$

$$x(2x+3)(4x+1)=0$$

$$x=0,-\frac{3}{2},-\frac{1}{4}$$

But $x>-1$ & $x\neq 0$

So, $x=-\frac{1}{4}$

85. $$f(x)=\int_0^4 e^{|x-t|}dt=\int_0^x e^{(x-t)}dt+\int_x^4 e^{(t-x)}dt=e^x+e^{4-x}-2\geq 2e^2-2$$

86. $$\frac{1}{4}\int_0^\infty\frac{4\cos^3 x}{x}dx=\frac{1}{4}\int_0^\infty\frac{\cos 3x+3\cos x}{x}dx=\frac{1}{4}\int_0^\infty\frac{\cos 3x}{x}dx+\frac{3}{4}\int_0^\infty\frac{\cos x}{x}dx$$

87. $$\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)dx=\int\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx=\int\cos x\,dx$$

88. $I_{n+\frac{1}{2}} = \int_0^{\pi} \frac{\sin(2nx + x)}{\sin 2x} dx = \int_0^{\pi} \frac{\sin 2nx \cdot \cos x}{\sin 2x} dx + \int_0^{\pi} \frac{\cos 2nx \cdot \sin x}{\sin 2x} dx$

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin 2nx}{\sin x} dx + \frac{1}{2}\int_0^{\pi} \frac{\cos 2nx}{\cos x} dx$$

89. $f'(x) = 1 + \ln^2 x + 2\ln x = 0 \Rightarrow x = \frac{1}{e}$

$f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_1^{1/e} (\ln^2 t + 2\ln t)\, dt$

Let $I = \int_1^{1/e} (\ln^2 t + 2\ln t)\, dt$

$\ln t = x \Rightarrow t = e^x; dt = e^x dx = \int_0^{-1} (x^2 + 2x)\, e^x dx = [e^x \cdot x^2]_0^{-1} = \frac{1}{e}$

90. $f(x) = x^2 + \int_0^x e^{-t} f(x-t)\, dt$

$x^2 + \int_0^x e^{t-x} f(t)\, dt = x^2 + e^{-x}\int_0^x e^t f(t)\, dt$

$\Rightarrow \quad f'(x) = 2x - e^{-x}\int_0^x e^t f(t)\, dt + f(x)$

$\Rightarrow \quad f'(x) = 2x + x^2 \Rightarrow f(x) = \frac{x^3}{3} + x^2$

$\Rightarrow \quad y = \frac{1}{4}(-2x^2 + 6x - 1)$

91. $I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+5^x} dx = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+5^{-x}} dx$

$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \cos^2 x\, dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}$

92. $\int\left(\frac{x^2-x+1}{x^2+1}\right)e^{\cot^{-1}x}dx$

Let $\cot^{-1}x=1 \Rightarrow \frac{-1}{1+x^2}dx=dt$

$\int e^t(\cot t-\operatorname{cosec}^2 t)\,dt=e^t\cdot\cot t+c$

$=x\cdot e^{\cot^{-1}x}+c$

93. $\lim\limits_{x\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r\sqrt{n^2+r^2}}{n^2}\right)=\lim\limits_{x\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\frac{r}{n}\sqrt{1+\left(\frac{r}{n}\right)^2}=\int\limits_0^1 x\sqrt{1+x^2}\,dx=\left[\frac{(1+x^2)^{3/2}}{3}\right]_0^1$

94. $\int\frac{(x^3-1)}{(x^4+1)(x+1)}dx=\int\frac{x^3}{x^4+1}dx-\int\frac{1}{x+1}dx=\frac{1}{4}\ln(1+x^4)-\ln(1+x)+c$

95. $\lim\limits_{x\to 0^+}\frac{(\cos^{-1}\cos x)(-\sin x)}{2-2\cos 2x}=\lim\limits_{x\to 0^+}\frac{-x\sin x}{4\sin^2 x}=-\frac{1}{4}$

96. $f(x)\begin{cases}0 & x>\tan 1\\ \cos x & 0<x<\tan 1\\ \frac{\cos x}{2} & x=\tan 1\end{cases}$

$\int\limits_0^\infty f(x)\,dx=\int\limits_0^{\tan 1}\cos x+\int\limits_{\tan 1}^{\infty}0\,dx=\sin(\tan 1)$

97. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\frac{k}{n^2+n+2k}\right)=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\left(\frac{\frac{k}{n}}{1+\frac{1}{n}+\frac{2k}{n^2}}\right)=\int\limits_0^1 x\,dx$

98. $\lim\limits_{y\to 1^+}\frac{\int_1^y|t-1|\,dt}{\tan(y-1)}\Rightarrow\lim\limits_{y\to 1^+}\frac{y-1}{\sec^2(y-1)}=0$ (Applying L' Hospital Rule)

99. $\int_0^1\frac{dx}{(1+x^2)^4}=\left[\frac{x}{2(4-1)(1+x^2)^{4-1}}\right]_0^1+\frac{5}{6}\int_0^1\frac{dx}{(1+x^2)^3}$

$=\left(\frac{1}{6(2)^3}-0\right)+\frac{5}{6}\left[\frac{x}{2(2)(1+x^2)^2}\right]_0^1+\frac{5}{6}\cdot\frac{3}{4}\int_0^1\frac{dx}{(1+x^2)^2}$

$$=\frac{1}{48}+\left(\frac{5}{6}\right)\left[\frac{1}{16}-0\right]+\frac{5}{8}\left[\frac{x}{2(1)(1+x^2)}\right]_0^1+\frac{5}{8}\times\frac{1}{2}\int_0^1\frac{dx}{1+x^2}$$

$$=\frac{1}{48}+\frac{5}{6\times 16}+\frac{5}{8}\left(\frac{1}{4}-0\right)+\frac{5}{16}[\tan^{-1}x]_0^1$$

$$=\frac{7}{6\times 16}+\frac{5}{8\times 4}+\frac{5}{16}\left[\frac{\pi}{4}-0\right]$$

$$=\frac{22}{6\times 16}+\frac{5\pi}{64}$$

$$=\frac{11}{48}+\frac{5\pi}{64}$$

Alternate solution :

$$I=\int_0^1\frac{dx}{(1+x^2)^4}$$

Put $x=\tan\theta$; therefore, $dx=\sec^2\theta\, d\theta$.

$$I=\int_0^{\pi/4}\frac{\sec^2\theta\, d\theta}{(\sec\theta)^8}$$

That is,

$$I=\int_0^{\pi/4}(\cos\theta)^6 d\theta$$

$$=\int_0^{\pi/4}\left(\frac{3\cos\theta+\cos 3\theta}{4}\right)^2 d\theta$$

$$=\frac{9}{16}\int_0^{\pi/4}\cos^2\theta\, d\theta+\frac{1}{16}\int_0^{\pi/4}(\cos 3\theta)^2 d\theta+\frac{3}{8}\int_0^{\pi/4}\cos\theta\cos 3\theta\, d\theta$$

$$=\frac{9}{16}\int_0^{\pi/4}\frac{1+\cos 2\theta}{2}d\theta+\frac{1}{16}\int_0^{\pi/4}\frac{1+\cos 6\theta}{2}d\theta+\frac{3}{8}\int_0^{\pi/4}\frac{\cos 4\theta+\cos 2\theta}{2}d\theta$$

$$=\frac{9}{32}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/4}+\frac{1}{16\times 2}\left[\theta+\frac{\sin 6\theta}{6}\right]_0^{\pi/4}+\frac{3}{8\times 2}\left[\frac{\sin 4\theta}{4}+\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$=\left(\frac{9}{32}\right)\left[\frac{\pi}{4}+\frac{1}{2}\right]+\frac{1}{16\times 2}\left[\frac{\pi}{4}-\frac{1}{6}\right]+\frac{3}{8\times 2}\left[0+\frac{1}{2}-0\right]$$

$$=\frac{5}{64\pi}+\frac{11}{48}$$

100. We have,
$$I=\int_0^{\pi/4}(\sin x)^4 dx \quad \ldots(1)$$

We know that, $\sin^2 x=\frac{1-\cos 2x}{2}$

Therefore,

$$\sin^4 x = (\sin x)^4 = \left(\frac{1-\cos 2x}{2}\right)^2$$

$$= \frac{1}{4}[1 - 2\cos 2x + (\cos 2x)^2]$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right)$$

Substituting this value of $\sin^4 x$ in Eq. (1), we get

$$I = \int_0^{\pi/4} \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \left[\frac{3}{8}x\right]_0^{\pi/4} - \frac{1}{4}[\sin 2x]_0^{\pi/4} + \frac{1}{32}[\sin 4x]_0^{\pi/4}$$

$$= \left(\frac{3}{8}\cdot\frac{\pi}{4}\right) - \frac{1}{4}(1-0) + \frac{1}{32}(0-0)$$

$$= \frac{3\pi}{32} - \frac{1}{4}$$

Alternate solution : We have,

$$I = \int_0^{\pi/4} (\sin x)^4 dx$$

which can be written as

$$J = \int (\sin^2 x)(1 - \cos^2 x)\, dx$$

$$= \int \sin^2 x\, dx - \frac{1}{4}\int 4\sin^2 x \cos^2 x\, dx$$

$$= \int \frac{1-\cos 2x}{2} dx - \frac{1}{4}\int (\sin 2x)^2 dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{1}{4}\int \frac{1-\cos 4x}{2} dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{8}x + \frac{1}{32}\sin 4x + c$$

$$= \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

Using the given limits, the above equation becomes

$$I = [J]_0^{\pi/4} = \left[\frac{3}{8}x\right]_0^{\pi/4} - \left[\frac{\sin 2x}{4}\right]_0^{\pi/4} + \left[\frac{\sin 4x}{32}\right]_0^{\pi/4}$$

$$= \frac{3\pi}{32} - \frac{1}{4}$$

101. $\int \frac{(\cos 9x + \cos 6x)\sin 5x}{\sin 10x - \sin 5x} dx = \int 2\cos\frac{5x}{2}\cos\frac{3x}{2} = \int(\cos 4x + \cos x)$

$$= \frac{\sin 4x}{4} + \sin x + C$$

$A = \frac{1}{4}, B = 1$

102. $\int \frac{\frac{dx}{x^{2014}}}{1 + \frac{1}{x^{2013}}} = \frac{1}{2013}\ln\left(\frac{x^{2013}}{1 + x^{2013}}\right) + C$

103. $\frac{1}{2}\int_0^1 x\cdot(2x\cdot e^{-x^2})\,dx = \frac{1}{2}\left[\left(-xe^{-x^2}\right) + \int_0^1 e^{-x^2}dx\right]$

$$= \frac{1}{2}\left[-\frac{1}{e} + a\right]$$

104. $2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx\right] + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$

$$= 2\left[\frac{0^2}{2} + \frac{1^2}{2} + \frac{2^2}{2}\right] + \frac{3^2}{2} + \frac{4^2}{2} = \frac{35}{2}$$

105. $\frac{1}{3}\int \frac{3x^2}{x^6(1+x^3)^2}dx$

Let $1 + x^3 = t \Rightarrow 3x^2dx = dt$

$\Rightarrow \quad \frac{1}{3}\int \frac{dt}{t^2(t-1)^2} = \frac{1}{3}\int\left(\frac{2}{t} + \frac{1}{t^2} - \frac{2}{t-1} + \frac{1}{(t-1)^2}\right)dt$

106. $\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{1}{n\sqrt{1+\frac{r}{n}}} = \int_0^3 \frac{1}{\sqrt{1+x}}dx = (2\sqrt{1+x})_0^3 = 2$

107. $\int_0^2 x\,f(x)\,dx = \left[\frac{x^2}{2}f(x)\right]_0^2 - \int_0^2 \frac{x^2}{2}f'(x)\,dx = 0 + \int_0^2 \frac{x^2}{2\sqrt{1+x^3}}dx$

108. $\int_0^{\pi/3}(\ln(\cos x+\sqrt{3}\sin x)-\ln\cos x)\,dx$

$$=\int_0^{\pi/3}\left\{\ln\left(2\cos\left(x-\frac{\pi}{3}\right)\right)-\ln\cos x\right\}dx=\frac{\pi}{3}\ln 2$$

109. $\sum_{r=1}^{100}\int_0^1 f(r-1+x)\,dx=\int_0^1 f(x)\,dx+\int_0^1 f(x+1)\,dx+\int_0^1 f(x+2)\,dx+\ldots+\int_0^1 f(x+99)\,dx$

$$=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx+\ldots+\int_{99}^{100} f(x)\,dx=\int_0^{100} f(x)\,dx=a$$

110. $\lim_{n\to\infty}\sum_{k=0}^{n}x^2\left(\frac{(2x)^k}{k!}\right)=x^2\cdot e^{2x}\Rightarrow\int_0^1 x^2e^{2x}dx=\frac{e^2-1}{4}$

111. $\int x^5\sqrt{1+x^3}\,dx$

Let $1+x^3=t^2$

$3x^2dx=2t\,dt$

$\frac{2}{3}\int t^2(t^2-1)\,dt=\frac{2}{3}\left(\frac{t^5}{5}-\frac{t^3}{3}\right)+c$

112. $f'(x)=\frac{\sin x}{x}$

$f'(x)>0\ \forall\ x\in(0,\pi)$

$f'(x)<0\ \forall\ x\in(\pi,2\pi)$

113. $\int\frac{x(x^2+1)+3(x^2+3)}{(x^2+1)(x^2+3)}dx$

$\int\left(\frac{x}{x^2+3}+\frac{3}{x^2+1}\right)dx$

$\frac{1}{2}\ln|x^2+3|+3\tan^{-1}x+c$

114. $\int\frac{\sqrt{\sec^5 x}}{\sqrt{\sin^3 x}}dx=\int\frac{\sec^4 x}{\sqrt{\tan^3 x}}dx$

Let $\tan x=t^2$

$\sec^2 x\,dx=2t\,dt$

$\int\frac{(1+t^4)\cdot 2t\cdot dt}{t^3}=2\int\left(\frac{1}{t^2}+t^2\right)dt$

115. Let $tx = y \Rightarrow x\,dt = dy$

$$\lim_{x\to 0}\frac{\int_0^{x^2} e^{\sin y}\,dy}{x^2} = \lim_{x\to 0}\frac{e^{\sin x^2}\,2x}{2x} = 1$$

116. $$\int_0^{\pi/2}\frac{\cos 2x}{x}\,dx = \left(\frac{\sin 2x}{2x}\right)_0^{\pi/2} + 2\int_0^{\pi/2}\frac{\sin 2x}{(2x)^2}\,dx = -1 + \int_0^{\pi}\frac{\sin\theta}{\theta^2}\,d\theta \qquad (\because \text{ Let } 2x = \theta)$$

Exercise-2 : One or More than One Answer is/are Correct

1. $\int \frac{dx}{(1+\sqrt{x})^8}$ Let $x = t^2 \Rightarrow dx = 2t\,dt$

$$\int \frac{2t\,dt}{(1+t)^8} = 2\left[\int\frac{dt}{(t+1)^7} - \int\frac{dt}{(t+1)^8}\right] = 2\left[-\frac{1}{6(1+t)^6} + \frac{1}{7(1+t)^7}\right] + C$$

2. $$\int_{-\alpha}^{\alpha}\left[e^x + \cos x \ln(x + \sqrt{1+x^2})\right]dx = 2\int_0^{\alpha} e^x\,dx = 2(e^{\alpha} - 1) \Rightarrow e^{\alpha} > \frac{7}{4}$$

3. $I = \int \sqrt{\frac{x}{a^3 - x^3}}\,dx$ Let $x^{3/2} = a^{3/2}\cos\theta$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\,dx = \frac{2}{3}\int \frac{a^{3/2}\cos\theta}{\sqrt{a^3 - a^3\sin^2\theta}}\,d\theta$$

$$= \frac{2}{3}\int d\theta = \frac{2}{3}\theta + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

4. $$\int x\sin x\sec^3 x\,dx = \int \underset{\text{I}}{x}\left(\underbrace{\tan x\sec^2 x}_{\text{II}}\right)dx = x\frac{\tan^2 x}{2} - \int\frac{\tan^2 x}{2}\,dx$$

$$= x\frac{\tan^2 x}{2} - \int\frac{(\sec^2 x - 1)}{2}\,dx = x\frac{\tan^2 x}{2} - \frac{1}{2}(\tan x - x) + C$$

$$= \frac{1}{2}(x\sec^2 x - \tan x) + C$$

$\therefore \quad f(x) = \sec^2 x, \quad g(x) = \tan x$

(a) Clear $f(x) \notin (-1, 1)$

(b) $\tan x = \sin x$

$\Rightarrow \quad \cos x = 1 \;\Rightarrow\; \tan x$ is not defined.

no solution

(c) $g'(x) = f(x)\ \forall\ x \in R$ except $(2n-1)\dfrac{\pi}{2}$

(d) $\sec^2 x = \tan x$

$1 + \tan^2 x - \tan x = 0$ has no solution.

5. $\int (\sin 3\theta + \sin\theta)\cos\theta\, e^{\sin\theta} d\theta = \int (4\sin\theta - 4\sin^3\theta)\, e^{\sin\theta}\cos\theta\, d\theta$

Let $t = \sin\theta$

$dt = \cos\theta\, d\theta = 4[-t^3 + 3t^2 - 5t + 5]e^t + C$

Compare it

$$A = -4, B = -12, C = -20$$

7. $I = \int_0^\theta \dfrac{2x\,dx}{\sqrt{(3\theta - 2x)(\theta + 2x)}} = \int_0^\theta \dfrac{2(\theta - x)\,dx}{\sqrt{(3\theta - 2x)(\theta + 2x)}}$

$\Rightarrow \quad I = \dfrac{\theta}{2}\int_0^\theta \dfrac{dx}{\sqrt{\theta^2 - (x - \theta/2)^2}}$

8. Let $f(x) = a^x, F(x) = F(-x)$

9. $J = \int_{-1}^{0}\left[\cot^{-1}\left(\dfrac{1}{x}\right) + \cot^{-1}(x)\right]dx + \int_0^2 \left(\cot^{-1}\left(\dfrac{1}{x}\right) + \cot^{-1} x\right)dx$

$= \int_{-1}^{0}\left(\pi + \dfrac{\pi}{2}\right)dx + \int_0^2 \dfrac{\pi}{2}\,dx$

$K = \int_0^\pi dx = \pi \qquad$ (As 2π is period)

11. $l_1 = \lim\limits_{x\to\infty}\sqrt{\dfrac{x - \cos^2 x}{x + \sin x}} = 1$

$l_2 = \lim\limits_{h\to 0^+}\int_{-1}^{1}\dfrac{h\,dx}{h^2 + x^2} = \lim\limits_{h\to 0} 2\tan^{-1}\dfrac{1}{h} = \pi$

13. $\int \dfrac{dx}{(1+\sin^2 x)\cos^2 x} = \int \dfrac{\sec^4 x}{1 + 2\tan^2 x}\,dx$

$$=\int \frac{(1+\tan^2 x)\sec^2 x}{(1+2\tan^2 x)}dx = \frac{1}{2}\int \sec^2 x\,dx + \frac{1}{2}\int \frac{\sec^2 x\,dx}{1+2\tan^2 x}$$

$$=\frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c$$

14. $\int \frac{(1+\sin^{2015} x)-\sqrt{1+\sin^{4030} x}}{2\sin^{2015} x}$ (Rationalise)

$\int_{-2014}^{2014} \frac{1}{2}dx$ $\qquad \int \text{odd} = 0$

15. $\tan^{-1}(nx)|_a^{\infty} = \frac{\pi}{2} - \tan^{-1}(na)$

$a>0, a=0, a<0$

16. Let $\sqrt{x} = \cos 2\theta$

$dx = -\sin 4\theta\, d\theta$

$I = \int_0^{\pi/4} \cot\theta \sin 4\theta\, d\theta$ and $J = \int_0^{\pi/4} \tan\theta \sin 4\theta\, d\theta$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

2. $f(x) = \int (2x^3\cos^2 x + 6x^2 \sin x \cos x - 2x^3 \sin^2 x)\,dx$

$$= \int \left(\underbrace{2x^3}_{\text{I}} \underbrace{\cos 2x}_{\text{II}} + 3x^2 \sin 2x \right) dx$$

$f(x) = x^3 \sin 2x + c$

$f(\pi) = 0 + c = 0 \Rightarrow c = 0$

$f(x) = x^3 \sin 2x$

Paragraph for Question Nos. 6 to 8

6. $g(x) = x - A$

$A = \int_0^1 f(t)\,dt$

$$f(x)=\frac{x^3}{2}+1-x\int_0^x(x-A)\,dx=\frac{x^3}{2}+1-\frac{x^3}{2}+Ax^2$$

$$f(x)=Ax^2+1$$

$$A=\int_0^1 Ax^2+1 \;\Rightarrow\; A=\frac{3}{2}$$

$$f(x)=\frac{3x^2}{2}+1;\quad \text{min. } f(x)=1$$

7. $\frac{3}{2}x^2+1=x-\frac{3}{2}$

$3x^2-2x+5=0$

$\Delta<0$, no solution

8. $g(x)=x-\frac{3}{2}$

$A=\frac{1}{2}\times\frac{3}{2}\times\frac{3}{2}=\frac{9}{8}$

Paragraph for Question Nos. 9 to 11

9. $$\int_0^a f(x)\,dx-\int_a^1 f(x)\,dx=2f(a)+3a+b \qquad ...(1)$$

Diff. w.r.t. 'a' on both sides,

$$(f(a)-0)-(0-f(a))=2f'(a)+3$$

$$2f(a)=2f'(a)+3$$

$$(2f(a)-3)=2f'(a)$$

$$\frac{2f'(a)}{2f(a)-3}=1$$

$$\int\frac{2f'(a)}{2f(a)-3}\,da=\int da$$

$$\ln|2f(a)-3|=a+c$$

$$2f(a)-3=e^{a+c}$$

$$2f(a)-3=ke^a$$

$$2f(a)=ke^a+3$$

Put $a=1 \Rightarrow 0=ke+3 \Rightarrow k=-\frac{3}{e}$

$$\therefore \quad 2f(a)=-\frac{3}{e}e^a+3$$

$$f(a)=\frac{3}{2}-\frac{3}{2e}e^a$$

$$f(x)=\frac{3}{2}-\frac{3}{2e}e^x$$

Put $f(x)$ in (1) (By taking limiting case)

$$\int_0^a\left(\frac{3}{2}-\frac{3}{2e}e^x\right)dx-\int_a^1\left(\frac{3}{2}-\frac{3}{2e}e^x\right)dx=3-\frac{3}{e}e^a+3a+b$$

$$\left[\left(\frac{3a}{2}-\frac{3}{2e}e^a\right)-\left(0-\frac{3}{2e}\right)\right]-\left[\left(\frac{3}{2}-\frac{3}{2}\right)-\left(\frac{3a}{2}-\frac{3e^a}{2e}\right)\right]=3-\frac{3}{e}e^a+3a+b$$

$$\frac{3}{2e}-3=b$$

10. Length of subtangent $=\left|\frac{y}{(dy/dx)}\right|$

$$y=f(x)=\frac{3}{2}-\frac{3}{2e}e^x$$

$$\frac{dy}{dx}=f'(x)=0-\frac{3}{2e}e^x$$

$$\left.\frac{dy}{dx}\right|_{x=1/2}=-\frac{3}{2\sqrt{e}}$$

when $x=\frac{1}{2}$, $y=f\left(\frac{1}{2}\right)=\frac{3}{2}-\frac{3}{2e}e^{1/2}=\frac{3}{2}\left(1-\frac{1}{\sqrt{e}}\right)$

$$\text{Length of subtangent}=\left|\frac{\frac{3}{2}\left(1-\frac{1}{\sqrt{e}}\right)}{-\frac{3}{2\sqrt{e}}}\right|=\left|\sqrt{e}-1\right|=\sqrt{e}-1$$

11. $\int_0^1 f(x)\,dx=\int_0^1\left(\frac{3}{2}-\frac{3}{2e}e^x\right)dx=\frac{3x}{2}-\frac{3}{2e}e^x\Big|_0^1=\left(\frac{3}{2}-\frac{3e}{2e}\right)-\left(0-\frac{3}{2e}\right)$

$$=\left(\frac{3}{2}-\frac{3}{2}\right)+\frac{3}{2e}=\frac{3}{2e}$$

Paragraph for Question Nos. 12 to 13

12. $f_3'''(x) = f_0(x)$ see options or 3 times by parts.

13. $f_n(x) = \dfrac{x^n}{\lfloor n}\left(\ln x - 1 - \dfrac{1}{2} - \dfrac{1}{3} \ldots - \dfrac{1}{n}\right)$

Paragraph for Question Nos. 14 to 15

$$f(x) = a\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$f(1) = 1 \qquad a = 1$

$f(x) = x^2 - x + 1$

14. $g(x) = 1 + x^2$ now integrate

15. $\displaystyle\int \frac{e^x}{e^{2x} - e^x + 1} \quad e^x = t$

Paragraph for Question Nos. 16 to 17

17. $$L = \lim_{x\to\infty} \frac{xe^{2x}(1+3x^2)^{1/2}}{C\cdot(xe^x)^{C-1}\cdot(e^x + xe^x)} = \lim_{x\to\infty} \frac{(xe^x)^2\left(\dfrac{1}{x^2}+3\right)^{1/2}}{C\cdot(xe^x)^C\cdot\left(\dfrac{1}{x}+1\right)}$$

Exercise-4 : Matching Type Problems

2. (A) $$\int \frac{dx}{(x^2+1)\sqrt{x^2+2}} = \int \frac{\dfrac{1}{x^3}dx}{\left(1+\dfrac{1}{x^2}\right)\sqrt{1+\dfrac{2}{x^2}}}$$

Let $1 + \dfrac{2}{x^2} = t^2$

(C) $$\int \frac{x^4 + x^8}{(1-x^4)^{7/2}} = \int \frac{\left(x + \dfrac{1}{x^3}\right)dx}{\left(\dfrac{1}{x^2} - x^2\right)^{7/2}}$$

Let $\dfrac{1}{x^2} - x^2 = t^2$

(D) Let $\sqrt{x} = \cos 2\theta \Rightarrow dx = -2\sin 4\theta\, d\theta$

$$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx = -2\int \tan\theta \sin 4\theta\, d\theta$$

3. (A) Let $\sin x = t \quad \Rightarrow \cos x\, dx = dt$

$$\int_0^1 \frac{dt}{(1+t)(2+t)} = \int_0^1 \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = [\ln(1+t) - \ln(t+2)]_0^1$$

(B) $$\int_0^{41\pi/4} |\cos x|\, dx = 10\int_0^{\pi} |\cos x|\, dx + \int_0^{\pi/4} \cos x\, dx$$

$$= 10\left[\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right] + \int_0^{\pi/4} \cos x\, dx$$

(C) $$\int_{-1/2}^{0} [x]\, dx + \int_0^{1/2} [x]\, dx + \int_{-1/2}^{1/2} \ln\left(\frac{1+x}{1-x}\right) dx$$

$$= \int_{-1/2}^{0} -1\, dx + \int_0^{1/2} 0\, dx = -\frac{1}{2}$$

(D) $$I = \int_0^{\pi/2} \frac{2\sqrt{\cos\theta}}{3(\sqrt{\cos\theta} + \sqrt{\sin\theta})}\, d\theta = \int_0^{\pi/2} \frac{2\sqrt{\sin\theta}}{3(\sqrt{\sin\theta} + \sqrt{\cos\theta})}\, d\theta$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{2}{3}\, d\theta = \frac{\pi}{3}$$

$$\Rightarrow \quad I = \frac{\pi}{6}$$

4. (A) Common root $\alpha = b - a \Rightarrow 3(b-a)^2 + a(b-a) + 1 = 0 \Rightarrow 2a^2 + 3b^2 - 5ab + 1 = 0$

(B) $$\frac{x^4+1}{2x^2} = \sin^2\frac{\pi x}{2} \Rightarrow \frac{x^2 + \frac{1}{x^2}}{2} = \sin^2\frac{\pi x}{2} \Rightarrow x = \pm 1$$

(C) $$y = \frac{1}{\frac{1}{(x-1)^2} + \frac{1}{x-1} - 2} \qquad x \neq 1 \;;\; \frac{1}{x-1} \neq -2, 1$$

(D) $$\int \left(\frac{x}{1+x}\right)^{7/6} \frac{dx}{x^2}.$$

Let $\frac{x+1}{x}=t^6 \Rightarrow -\frac{1}{x^2}dx=6t^5dt$

$\therefore \quad I=6\int t^{-7}(-t^5)\,dt=\frac{6}{t}+C=6\left(\frac{x}{x+1}\right)^{1/6}+C$

5. (A) We have,

$$\int_0^{1.5}[x^2]\,dx=\int_0^1[x^2]\,dx+\int_1^{\sqrt{2}}[x^2]\,dx+\int_{\sqrt{2}}^{1.5}[x^2]\,dx$$

$$=\int_0^1 0\,dx+\int_1^{\sqrt{2}}1\cdot dx+\int_{\sqrt{2}}^{1.5}2\cdot dx=0+(\sqrt{2}-1)+2(1.5-\sqrt{2})=2-\sqrt{2}$$

(B) We have,

$$\int_0^4\{\sqrt{x}\}\,dx=\int_0^1\sqrt{x}\,dx+\int_1^4(\sqrt{x}-1)\,dx=\frac{2}{3}+\frac{2}{3}(8-1)-3=\frac{7}{3}$$

Aliter : $\int_0^4\{\sqrt{x}\}\,dx=\int_0^4\sqrt{x}\,dx-\int_0^4[\sqrt{x}]\,dx$

(C) We have,

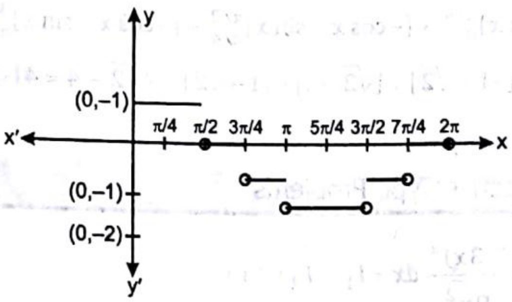
$$\sin x+\cos x=\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)$$

$$\therefore \quad [\sin x+\cos x]=\left[\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right]$$

The graph of $y=\left[\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right]$ is obtained from the graph of $y=[\sqrt{2}\sin x]$ by translating it by $\frac{\pi}{4}$ units in the direction of OX'. The graph so obtained is shown in figure.

It is evident from the graph of $y=[\sqrt{2}\sin(x+\pi/4)]$ that

$$f(x)=[\sin x+\cos x]=\begin{cases}1, & 0\le x\le \pi/2\\ 0, & \pi/2<x\le 3\pi/4\\ -1, & 3\pi/4<x\le\pi\\ -2, & \pi<x<3\pi/2\\ -1, & 3\pi/2\le x<7\pi/4\\ 0, & 7\pi/4\le x<2\pi\end{cases}$$



$\therefore \quad \int_0^{2\pi} [\sin x + \cos x]\, dx$

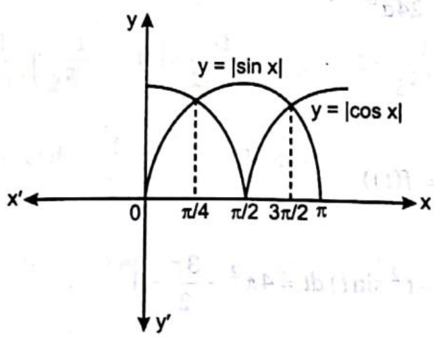
$$= \int_0^{\pi/2} 1 \cdot dx + \int_{\pi/2}^{3\pi/4} 0\, dx + \int_{3\pi/4}^{\pi} (-1)\, dx + \int_{\pi}^{3\pi/2} (-2)\, dx + \int_{3\pi/2}^{7\pi/4} (-1)\, dx + \int_{7\pi/4}^{2\pi} 0\, dx$$

$$= \frac{\pi}{2} + 0 - \frac{\pi}{4} - 2 \times \frac{\pi}{2} + (-1)\frac{\pi}{4} + 0 = -\pi$$

(D) We have,

$$\int_0^{\pi} \big||\sin x| - |\cos x|\big|\, dx$$

$$= \int_0^{\pi/4} (|\sin x| - |\cos x|)\, dx + \int_{\pi/4}^{3\pi/4} (|\sin x| - |\cos x|)\, dx + \int_{3\pi/4}^{\pi} -(|\sin x| - |\cos x|)\, dx$$



$$= -\int_0^{\pi/4} (\sin x - \cos x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx + \int_{\pi/2}^{3\pi/4} (\sin x + \cos x)\, dx$$

$$+ \int_{3\pi/4}^{\pi} -(\sin x + \cos x)\, dx$$

$$= -[-\cos x - \sin x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} + [-\cos x - \sin x]_{\pi/2}^{3\pi/4} - [-\cos x + \sin x]_{3\pi/4}^{\pi}$$

$$= -[-\sqrt{2}+1] + [-1+\sqrt{2}] + [\sqrt{2}-1] - [1-\sqrt{2}] = 4\sqrt{2} - 4 = 4(\sqrt{2}-1)$$

Exercise-5 : Subjective Type Problems

1. $\int \frac{x\,dx}{\sqrt{1-9x^2}} + \int \frac{(\cos^{-1} 3x)^2}{\sqrt{1-9x^2}} dx = I_1 + I_2$

$I_1 = \int \frac{x\,dx}{\sqrt{1-9x^2}}$ $\qquad$ $I_2 = \int \frac{(\cos^{-1} 3x)^2}{\sqrt{1-9x^2}} dx$

Let $1-9x^2 = t^2$ $\qquad$ Let $\cos^{-1} 3x = k$

2. $I = \int_0^{\infty} \frac{x^3\,dx}{(a^2+x^2)^5}$

$I = \frac{1}{a^6}\int_0^{\pi/2} \sin^3\theta\cos^5\theta\,d\theta = \frac{1}{a^6}\int_0^{\pi/2}\cos^3\theta\sin^5\theta\,d\theta$ (Let $x = a\tan\theta$)

$2I = \frac{1}{8a^6}\int_0^{\pi/2}\sin^3 2\theta\,d\theta = \frac{1}{32a^6}\int_0^{\pi/2}(3\sin 2\theta - \sin 6\theta)\,d\theta$

$\Rightarrow \qquad I = \frac{1}{24a^6}$

3. $\int_0^{2\pi} g(x)\,dx$

$\int_{3\pi/2}^{2\pi} t\,f'(t)\,dt \qquad (\because x = f(t))$

$= \int_{3\pi/2}^{2\pi}(t\cos t - t^2\sin t)\,dt = 4\pi^2 - \frac{3\pi}{2} - 1$

4. $\int (x^5+x^3+x)\sqrt{2x^6+3x^4+6x^2}\,dx$

Let $2x^6+3x^4+6x^2 = t^2 \Rightarrow 12(x^5+x^3+x)\,dx = 2t\,dt$

$= \frac{1}{12}\int 2t^2 dt = \frac{1}{18}(2x^6+3x^4+6x^2)^{3/2} + C$

5. Put $x = \sin\theta$

6. $\int \frac{\tan x}{\tan^2 x + \tan x + 1} dx$

Let $\tan x = t, \sec^2 x\, dx = dt, dx = \frac{dt}{1+t^2}$

$$\int \frac{t}{(1+t+t^2)} \cdot \frac{dt}{1+t^2} = \int \left(\frac{1}{1+t^2} - \frac{1}{1+t+t^2}\right) dt = \int \frac{dt}{1+t^2} - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \tan^{-1}(t) - \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$$

$$= \tan^{-1}(\tan x) - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + C$$

7. Let $x^4 = t$

$4x^3 dx = dt$

$$\int_0^1 \frac{1+t^{2010}}{(1+t)^{2012}} dt = \int_0^1 \frac{1}{(1+t)^{2012}} dt + \int_0^1 \frac{1}{t^2\left(1+\frac{1}{t}\right)^{2012}} dt = \left.\frac{(1+t)^{-2011}}{-2011}\right|_0^1 + \left.\frac{\left(1+\frac{1}{t}\right)^{-2011}}{2011}\right|_0^1$$

$$\frac{-1}{2011}\left(\frac{1}{2^{2011}} - 1\right) + \frac{1}{2011}\left(\frac{1}{2^{2011}} - 0\right) = \frac{-1}{2011}\left(\frac{1}{2^{2011}} - 1 - \frac{1}{2^{2011}}\right) = \frac{1}{2011} = \frac{\lambda}{\mu}$$

8. $\int_1^{\sqrt{3}} (x^{2x^2} x + 2x^{2x^2} \cdot x \ln x)\, dx = \int_1^{\sqrt{3}} x^{2x^2}(x + 2x\ln x)\, dx$

$\int_1^{\sqrt{3}} \left(x^{x^2}\right)^2 (x + 2x\ln x)\, dx$ Let $x^{x^2} = t \Rightarrow x^2 \ln x = \ln t$; $(2x\ln x + x)\, dx = \frac{dt}{t}$

$$\int_1^{(\sqrt{3})^3} t^2 \cdot \frac{dt}{t} = \int_1^{(\sqrt{3})^3} t\, dt = \left.\frac{t^2}{2}\right|_1^{3^{3/2}} = \frac{3^3 - 1}{2} = 13$$

9. $\int \frac{dx}{(\cos x - \sin x)(1 + \sin x \cos x)} = 2\int \frac{(\cos x - \sin x)\, dx}{(\cos x - \sin x)^2 (2 + (\sin x + \cos x)^2 - 1)}$

$$= 2\int \frac{(\cos x - \sin x)\, dx}{((\sin^2 x + \cos^2 x) - 2\sin x \cos x)(1 + (\sin x + \cos x)^2)}$$

$$=2\int \frac{(\cos x-\sin x)\,dx}{(2-(\sin x+\cos x)^2)(1+(\sin x+\cos x)^2)}$$

$$=2\int \frac{dt}{(2-t^2)(1+t^2)} \text{ where } t=\sin x+\cos x$$

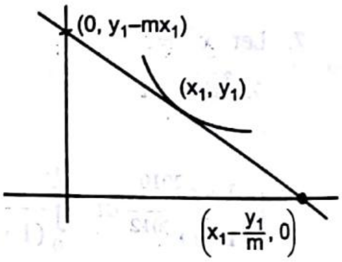
$$=2\int \frac{dt}{(2-t^2)(1+t^2)}=\frac{2}{3}\left[\int \frac{1}{1+t^2}dt+\int \frac{dt}{2-t^2}\right]$$

$$=\frac{2}{3}\left[\tan^{-1}(t)+\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+t}{\sqrt{2}-t}\right)\right]+C$$

$\therefore \quad A=\dfrac{2}{3},\ B=\dfrac{1}{3\sqrt{2}}$

$\therefore \quad 12A+9\sqrt{2}B-3=12\cdot\dfrac{2}{3}+9\sqrt{2}\,\dfrac{1}{3\sqrt{2}}-3=8$

10. $x^a\cdot y=\lambda^a;\quad y=\dfrac{\lambda^a}{x^a}$



$$\frac{dy}{dx}=-a\lambda^a x^{-a-1}=-a\frac{x^a\cdot y}{x^{a+1}}$$

$\Rightarrow \quad m=\dfrac{-ay_1}{x_1}$

$$A=\frac{1}{2}|y_1-mx_1|\left|x_1-\frac{y_1}{m}\right|=\frac{1}{2}y_1x_1(1+a)^2$$

$$=\frac{1}{2}\lambda^a\cdot x_1^{1-a}(1+a)^2$$

For A to be constant $1-a=0$

11. $I_{(6,8)}=\displaystyle\int_0^{\pi} x^6(\pi-x)^8\,dx=\left(-\frac{x^6(\pi-x)^9}{9}\right)_0^{\pi}+\int_0^{\pi}6x^5\frac{(\pi-x)^9}{9}dx$

$I_{(6,8)}=\dfrac{6}{9}I_{(5,9)}=\dfrac{6}{9}\times\dfrac{5}{10}\times\dfrac{4}{11}\times\dfrac{3}{12}\times\dfrac{2}{13}\times\dfrac{1}{14}\displaystyle\int_0^{\pi}(\pi-x)^{14}dx=\dfrac{6!\times 8!}{15!}\pi^{15}$

14. $I=\displaystyle\int_0^{100}\sqrt{x}\,dx-\left[\int_0^{1^2}0\cdot dx+\int_{1^2}^{2^2}dx+\ldots\ldots+\int_{2^2}^{3^2}2dx+\ldots\ldots+\int_{9^2}^{10^2}9\,dx\right]$

$I=\dfrac{155}{3}$

17. $f(\theta)=\int_{-1}^{1}\frac{\sin\theta\,dx}{(x-\cos\theta)^2+\sin^2\theta}=\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right)\Bigg|_{-1}^{1}$

Clearly, $f(\theta)$ is not defined when $\sin\theta=0$

$\theta=0,\pi,2\pi$

20. $f(x)=\frac{1}{2}\int_0^x(x-t)^2g(t)\,dt=\frac{1}{2}\left[x^2\int_0^x g(t)\,dt-2x\int_0^x tg(t)\,dt+\int_0^x t^2g(t)\,dt\right]$

$f'(x)=\left[x\int_0^x g(t)\,dt-\int_0^x t\,g(t)\,dt\right]$

$f''(x)=\int_0^x g(t)\,dt$

$f'''(x)=g(x)$

22. $f(2-x)=f(2+x)$, it means it symmetric about $x=2 \Rightarrow \int_0^2 f(x)\,dx=\int_2^4 f(x)\,dx=5$

Let $2-x=t$; $f(t)=f(4-t)$ *i.e.*, $f(x)=f(4-x)=f(4+x)$

$\int_0^{50} f(x)\,dx=25\left(\int_0^2 f(x)\,dx\right)=25\times5=125$

23. $I_n=\int_{-1}^{1}|x|\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots+\frac{x^{2n}}{2n}\right)dx=2\left[\int_0^1\left(x+\frac{x^3}{2}+\frac{x^5}{4}+\ldots\ldots+\frac{x^{2n+1}}{2n}\right)dx\right]$

$=2\left[\frac{x^2}{1\cdot2}+\frac{x^4}{2\cdot4}+\frac{x^6}{4\cdot6}+\ldots\ldots+\frac{x^{2n+1}}{2n\cdot(2n+2)}\right]_0^1$

$=2\left[\frac{1}{1\cdot2}+\frac{1}{2\cdot4}+\frac{1}{4\cdot6}+\ldots\ldots+\frac{1}{2n\cdot(2n+2)}\right]$

$I_n=1+\frac{1}{2}\left(\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\ldots\ldots+\frac{1}{n(n+1)}\right)=1+\frac{1}{2}\left(\frac{1}{1}-\frac{1}{n+1}\right)$

$\lim_{n\to\infty}I_n=1+\frac{1}{2}=\frac{3}{2}=\frac{p}{q}$

$pq(p+q)=3\times2(5)=30$

25. $\int_a^b|\sin x|\,dx=8 \quad \Rightarrow b-a=4\pi$

$$\int_0^{a+b} |\cos x|\, dx = 9 \quad \Rightarrow a+b = \frac{9\pi}{2} \Rightarrow a = \frac{\pi}{4}; b = \frac{17\pi}{4}$$

$$\frac{1}{\sqrt{2}\pi}\left|\int_a^b x\sin x \cdot dx\right| = \frac{1}{\sqrt{2}\pi}\left|\int_{\pi/4}^{17\pi/4} x\sin x\, dx\right| = 2$$

28. $f(x) = 0 \Rightarrow \int_0^x e^{-y} f'(y)\, dy = x^2 - x + 1$

$$\Rightarrow e^{-x} f'(x) = 2x - 1 \Rightarrow f(x) = (2x-3)e^x$$

29. $I_n = 2\int_0^{\pi}\left(\frac{\pi}{2} - |x|\right)\cos nx\, dx = 2\left(\frac{1-\cos n\pi}{n^2}\right)$

$$I_1 + I_2 + I_3 + I_4 = 4\left(1 + \frac{1}{9}\right) = \frac{40}{9}$$

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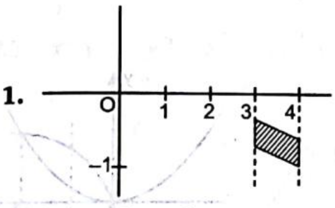
Chapter 6 – Area Under Curve

6 AREA UNDER CURVES

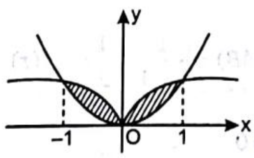
Exercise-1 : Single Choice Problems

1. 

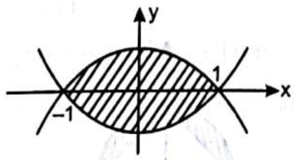
$1 \le x + 3y < 2$

Area of shaded region $= \frac{1}{3}$

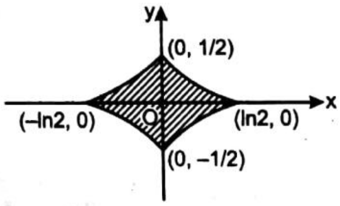
2. Area $= 2\int_0^1 (\sqrt{x} - x^2)\, dx$



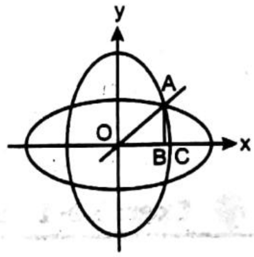
3. $y^2 = (x^2 - 1)^2 = 4\int_0^1 (1 - x^2)\, dx = \frac{8}{3}$



4. Area $= 4\int_0^{\ln 2} \left(e^{-x} - \frac{1}{2}\right) dx = 2 - 2\ln 2$



5. As given relations are inverse of each other so A lies on $y = x$

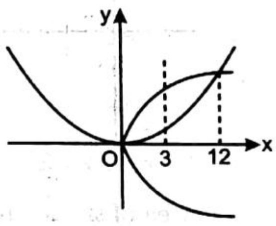


i.e., $\left(\frac{n}{\sqrt{n^2+1}}, \frac{n}{\sqrt{n^2+1}}\right)$

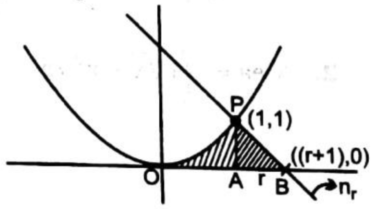
So, required area $= 8$ area $(OACBO) = 8\,(\Delta OAB + \text{area } BACB)$

$$= 8\left(\frac{1}{2}\left(\frac{n}{\sqrt{n^2+1}}\right)^2 + \int_{n/\sqrt{n^2+1}}^{1} n\cdot\sqrt{1-x^2}\,dx\right)$$

6. $$\frac{\int_0^3\left(12x - \frac{x^2}{12}\right)dx}{\int_3^{12}\left(12x - \frac{x^2}{12}\right)dx} = \frac{15}{49}$$



8. $\left.\frac{dy}{dx}\right|_{(1,1)} = r$

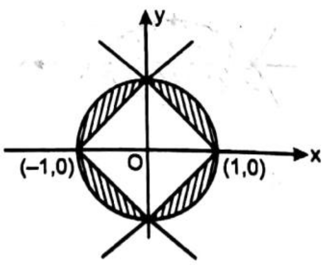


Normal at (1, 1) is $\Rightarrow y = -\frac{1}{r}(x-1)+1$

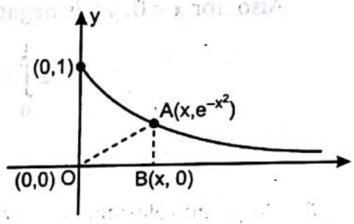
Required area $= \int_0^1 x^r\,dx + \text{ar}\,(\Delta PAB) = \frac{1}{r+1} + \frac{1}{2}r = f(r)$

$$f'(r) = -\frac{1}{(r+1)^2} + \frac{1}{2} = 0$$

9. Area $= 4\left[\int_0^1 \sqrt{1-x}\,dx - \frac{1}{2}\right] = \frac{2}{3}$



10. Area of $\Delta AOB = \frac{1}{2} x e^{-x^2}$



$$\frac{dA}{dx} = (1 - 2x^2)\, e^{-x^2}$$

Area is maximum at $x = \frac{1}{\sqrt{2}}$

12. $\int_1^2 g(x)\, dx$

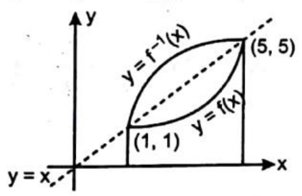
Let $x = f(t) \Rightarrow dx = f'(t)\, dt$

$$\int_0^1 t(3t^2 - 6t + 3)\, dt = \frac{1}{4}$$

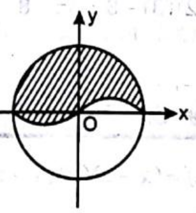
13. $x^2y^2 + y^4 - x^2 - 5y^2 + 4 = 0$

$(x^2 + y^2 - 4)(y^2 - 1) = 0$

14.



15. Ar. of shaded region $= \frac{1}{2}$ Ar. of circle $= \frac{\pi^3}{2}$.



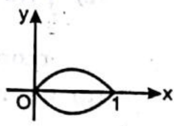
16. We have,

$$y^2 = x(1 - x^3) \quad \text{...(1)}$$

For $x > 1$, y^2 is negative. Since the square of a real number cannot be negative, y does not exist at $x = 0$ or at $x = 1$; $y = 0$. Let $x = \frac{1}{2}$. Therefore, from Eq. (1), we get

$$y^2 = \frac{1}{2}\left(1 - \frac{1}{8}\right) = \frac{7}{16}$$

$$y = \pm\frac{\sqrt{7}}{4}$$

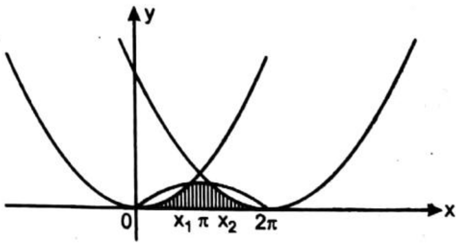


Also, for $x < 0$, y^2 is negative. Therefore, the required area is

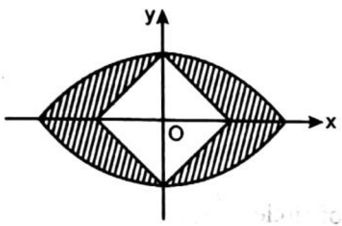
$$2\int_0^1 y\,dx = 2\int_0^1 (+)\sqrt{x}\sqrt{1-x^3}\,dx$$

$$= 2\int_0^1 \sqrt{x - x^4}\,dx$$

17.



18. $|x| + |y| \geq 2$ and $\frac{x^2}{9} + \frac{y^2}{4} = 1$



Ar. of ellipse – Ar. of square $= \pi(2)(3) - 8 = 6\pi - 8$

Exercise-2 : One or More than One Answer is/are Correct

1. (a) $f(x) = -(x-a)(x-b)(x+c) = (x-a)(x)(x+c)$ $[\because b = 0]$

Clearly option (a) is correct.

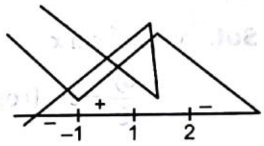
(b) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx > 0$ (from graph)

which incorrect

(c) $\int_a^b f(x)\,dx < 0$ & $\int_c^b f(x)\,dx < 0$ but second term is large negative value so option (c) is incorrect.

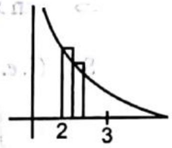
(d) Clearly, (d) is incorrect.

2. $T_n = \frac{1}{n}\sum_{r=2n}^{3n-1} \frac{(r/n)}{1+(r/n)^2}$, $S_n = \frac{1}{n}\sum_{r=2n+1}^{3n} \frac{(r/n)}{1+(r/n)^2}$



Let $f(x) = \frac{x}{1+x^2}$

$f'(x) = \frac{(1+x^2)-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$



$\therefore$ $f(x)$ is decreasing in (2, 3).

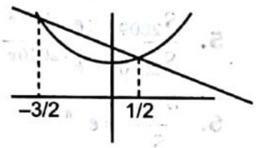
$T_n > \int_2^3 f(x)\,dx, \quad S_n < \int_2^3 f(x)\,dx$

3. $a + b = 2$ ...(1)

$\int_0^4 (a\sqrt{x} + bx)\,dx = 8 \quad \Rightarrow \frac{2a}{3} + b = 1$ ...(2)

4. Normal $y + x = \frac{7}{4}$

$\int_{-3/2}^{1/2} \left(\frac{7}{4} - x\right) - (x^2 + 1)\,dx$



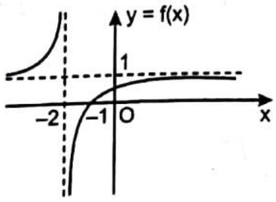
Exercise-3 : Comprehension Type Problems

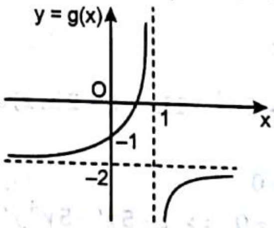
Paragraph for Question Nos. 1 to 3

Sol. $f(x) = \frac{x+a}{bx^2 + cx + 2}$

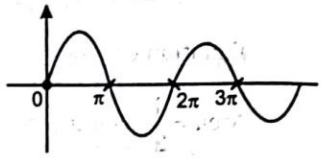
$f(-1) = 0 \Rightarrow a = 1$

If $y = 1$ is asymptotes, then $b = 0$ and $c = 1 \Rightarrow f(x) = \frac{x+1}{x+2}$ and $g(x) = \frac{1-2x}{x-1}$





Paragraph for Question Nos. 4 to 6



Sol. $y = e^{-x}\sin x$

$$\frac{dy}{dx} = e^{-x}[\cos x - \sin x] = 0$$

$$\Rightarrow \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$$

So, $(i.e., 0 \le x \le \pi)$

$$I = \int e^{-x}\sin x\,dx = \frac{-e^{-x}}{2}[\sin x + \cos x] + c$$

$$S_j = \left|\int_{j\pi}^{(j+1)\pi} e^{-x}\sin x\,dx\right| = \left|-\frac{e^{-x}}{2}[\sin x + \cos x]\right|_{j\pi}^{(j+1)\pi} = \frac{e^{-j\pi}}{2}(e^{-\pi}+1)$$

4. Put $j = 0, S_0 = \dfrac{1+e^{-\pi}}{2}$

5. $\dfrac{S_{2009}}{S_{2010}} = \dfrac{e^{-2009\pi}}{e^{-2010\pi}} = e^{\pi}$

6. $\dfrac{S_{j+1}}{S_j} = e^{-\pi}$

$$\because \sum_{j=0}^{\infty} S_j = \frac{S_0}{1-e^{-\pi}} = \frac{\frac{1+e^{-\pi}}{2}}{1-e^{-\pi}} = \frac{1+e^{\pi}}{2(e^{\pi}-1)}$$

Exercise-5 : Subjective Type Problems

1. $f(x) = x^2$

$$A = 2\int_0^1 (\sqrt{2-x^2} - x^2)\,dx = \frac{\pi}{2} + \frac{1}{3}$$

2. $f(x) = 2\ln x$

$$A = \int_0^1 (-x^3 + 6x^2 - 11x + 6 - 2\ln x)\,dx = \frac{17}{4}$$

4. At $x = 0, y = 0$

$x + 5y - y^5 = 0 \Rightarrow 1 + 5y' - 5y^4 y' = 0$

at $x = 0, y = 0$

$$y' = -\frac{1}{5}$$

Equation of tangent : $y = -\frac{x}{5}$, equation of normal : $y = 5x$

Area $= \frac{1}{2} \times 5 \times 26 = 65$

5. $[x]^2 = [y]^2$

$\therefore \quad [y] = \pm [x]$

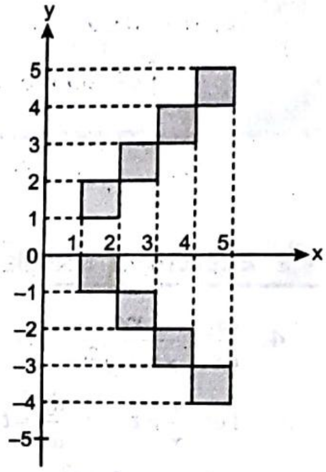
$[y] = \pm 1, \quad 1 \le x < 2$

$= \pm 2, \quad 2 \le x < 3$

$= \pm 3, \quad 3 \le x < 4$

$= \pm 4, \quad 4 \le x < 5$

$= \pm 5, \quad x = 5$



Now, when, $x \in [1, 2)$

then $y \in [-1, 0) \cup [1, 2)$

when $x \in [2, 3]$

then $y \in [-2, -1) \cup [2, 3)$

when $x \in [3, 4)$

then $y \in [-3, -2) \cup [3, 4)$

when $x \in [4, 5)$

then $y \in [-4, -3) \cup [4, 5)$

when $x = 5$

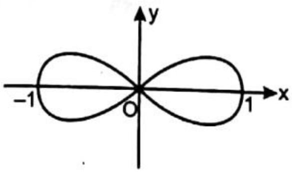
then $y \in [-5, -4) \cup [5, 6)$

Hence, required area $= 2(4) = 8$ sq. unit

6. $f(x) + f(z) = f(x + z)$ and $f(0) = 0$ and $f'(0) = 4 \Rightarrow f(x) = 4x$

So, area bounded $= \Delta = \int_0^4 (4x - x^2)\, dx$

7. Required area $= 4\int_0^1 \sqrt{x^2 - x^6}\, dx = \int_0^1 4x\sqrt{1 - x^4}\, dx = \frac{\pi}{2}$



Put $x^2 = \sin\theta$

8. $\text{Ar} = 4\int_0^1 (1 - x^{2/5})\, dx = 4\left(x - \frac{5}{7}x^{7/5}\right)_0^1 = \frac{8}{7}$

Chapter 7 – Differential Equations

7 DIFFERENTIAL EQUATIONS

Exercise-1 : Single Choice Problems

4. $\dfrac{dy}{dx}=\dfrac{xy}{x^2+y^2}$

Let $\dfrac{y}{x}=t \Rightarrow \dfrac{dy}{dx}=t+x\dfrac{dt}{dx}$

$\int\dfrac{dx}{x}=-\int\dfrac{1+t^2}{t^3}dt;\quad \ln y=\dfrac{x^2}{2y^2}-\dfrac{1}{2}\qquad (\because y(1)=1)$

5. $\int\dfrac{y}{\sqrt{1-y^2}}dy=\int dx$

$-\sqrt{1-y^2}=x+c$

$\Rightarrow\ (x+c)^2+y^2=1$

6. $\int\dfrac{dy}{y}=-\int\dfrac{dx}{(x-3)^2}$

$\Rightarrow\ \ln y=\dfrac{1}{x-3}+c$

7. Let $f(x)=y$

$\dfrac{dy}{dx}-2xy=\dfrac{e^{x^2}}{(x+1)^2}$

8. Let $x^2y^2=t$

$2xy^2+2x^2y\dfrac{dy}{dx}=\dfrac{dt}{dx}$

$\dfrac{dt}{dx}=\tan t$

9. $y = (C_1 \cos C_2)\cos x + (C_5 - C_1 \sin C_2)\sin x - C_3 e^{C_4} e^{-x}$

$y = A\cos x + B\sin x + Ce^{-x}$

$\because$ C_1, C_2, C_3, C_4 are arbitrary constants.

10. $y = e^{(\alpha+1)x}$

$y' = e^{(\alpha+1)x}(\alpha + 1)$

$y'' = e^{(\alpha+1)x}(\alpha + 1)^2$

12. $\frac{dy}{dx} - \left(1 + \frac{f'(x)}{f(x)}\right)y = f(x)$

$\text{I.F.} = e^{-\int\left(1+\frac{f'(x)}{f(x)}\right)dx} = \frac{e^{-x}}{f(x)}$

$\frac{ye^{-x}}{f(x)} = \int e^{-x}dx + C \quad \Rightarrow \frac{ye^{-x}}{f(x)} = -e^{-x} + C$

13. Equation of tangent at $\left(t, \frac{t^2}{2}\right)$ is

$$y = tx - \frac{t^2}{2} \quad \Rightarrow t = \frac{dy}{dx}$$

Differential equation is

$$\left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx} + 2y = 0$$

14. Let $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$; $\frac{1}{t^3}\frac{dt}{dx} + \frac{x}{t^2} = x \Rightarrow \frac{e^{-x^2}}{(x+y)^2} = e^{-x^2} + C$

15. $\frac{dy}{dx} - 2y\tan x = \tan^2 x$

$\text{I.F.} = e^{-2\int \tan x\, dx} = \cos^2 x$

$y\cos^2 x = \int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$

16. $f(x) = 2e^x + 1$

17. Let $\frac{dy}{dx} = t$ then $\frac{d^2y}{dx^2} = \frac{dt}{dx}$

$\frac{dt}{dx} = \frac{2tx}{x^2+1}$

$\Rightarrow \quad t = \frac{dy}{dx} = 3(x^2 + 1) \qquad (\because y'(0) = 3)$

$\Rightarrow \quad y = x^3 + 3x + 1 \qquad (\because y(0) = 1)$

18. $cy^2 = 2x + c$

$2cyy' = 2 \Rightarrow c = \dfrac{1}{yy'}$

$y^2 = 2x\,yy' + 1$

19. Let $\operatorname{cosec} y = t$

$\Rightarrow \quad -\operatorname{cosec} y \cot y\, dy = dt$

$\dfrac{dt}{dx} - \dfrac{t}{x} = -\dfrac{1}{x^2}$

$\Rightarrow \quad \dfrac{t}{x} = \dfrac{1}{2x^2} + c \Rightarrow \dfrac{1}{x \sin y} = \dfrac{1}{2x^2} + c$

20. $\dfrac{xdy - ydx}{x^2} = \dfrac{\sqrt{x^2+y^2}}{x^2}dx$

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

21. $\lim\limits_{t \to x} \dfrac{t^3 f(x) - x^3 f(t)}{t^2 - x^2} = \dfrac{1}{2} \Rightarrow 3x\,f(x) - x^2 f'(x) = 1$

$\Rightarrow \quad \dfrac{dy}{dx} - \dfrac{3y}{x} = \dfrac{-1}{x^2} \Rightarrow y = \dfrac{1}{4x} + \dfrac{3}{4}x^2 \qquad (\because f(1) = 1)$

22. $\dfrac{2dp(t)}{dt} = p(t) - 900$

$2\displaystyle\int \frac{dp(t)}{p(t) - 900} = \int dt$

$2 \ln |900 - p(t)| = t + c$

$p(t) = 900 - 50e^{t/2} \qquad (\because p(0) = 850)$

$p(t) = 0 \Rightarrow t = 2 \ln 18$

23. $\dfrac{\sin y}{\cos^2 y}\dfrac{dy}{dx} + \dfrac{\tan x}{\cos y} = \sec x$

Let $\dfrac{1}{\cos y} = t \Rightarrow \dfrac{\sin y}{\cos^2 y}dy = dt$

$\dfrac{dt}{dx} + t \cdot \tan x = \sec x$

$$t \cdot \sec x = \int \sec^2 x \cdot dx + C$$

$$\sec y \cdot \sec x = \tan x + C$$

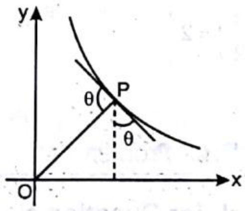
24. $\frac{dy}{dx} = (4x + y + 1)^2$

Let $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dt}{dx} = t^2 + 4$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + C$$

25. $\tan\theta = \left|\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x} \cdot \frac{dy}{dx}}\right| = -\frac{dx}{dy}$



$$\Rightarrow \left(\frac{dy}{dx}\right)^2 - \frac{2y}{x}\frac{dy}{dx} = 1$$

26. $\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$y \cdot x = \int x^3 dx = \frac{x^4}{4} + c$$

27. $x^3 dy + 3x^2 y\, dx = y^2 dx + 2xy\, dy$

$$d(x^3 y) = d(xy^2)$$

$$\int d(x^3 y) = \int d(xy^2) \Rightarrow x^3 y = xy^2$$

Exercise-2 : One or More than One Answer is/are Correct

1. $\frac{xdy - ydx}{x^2} = \frac{x^2 - 2}{x^2} dx$

$$\int d\left(\frac{y}{x}\right) = \int \left(1 - \frac{2}{x^2}\right) dx$$

$y = x^2 - 2x + 2$ $(\because f(1) = 1)$

2. I.F. $= x\sec x$; $yx\sec x = \tan x + c$

3. Put $y = h$

$\Rightarrow \quad x[f(x+h) - f(x-h)] - h[f(x+h) + f(x-h)] = 2(x^2 h - h^3)$

or $\displaystyle \lim_{h\to 0} x\frac{[f(x+h) - f(x-h)]}{h} - [f(x+h) + f(x-h)] = \lim_{h\to 0} 2(x^2 - h^2)$

$\Rightarrow \quad xf'(x) - f(x) = x^2 \quad \Rightarrow f(x) = x^2 + x$

4. L.D.E., I.F. $= 1 + \sin^2 x$; $(1 + \sin^2 x)f = \sin x + C$, $C = 0$

5. $2ydx + 2xdy + (2x^2 y^{3/2}dx + x^3 y^{1/2}dy) = 0$

$$2d(xy) + \frac{2}{3}d(x^3 \times y^{3/2}) = 0$$

6. I.F. $= \dfrac{1}{\sin^3 x}$; $\dfrac{y}{\sin^3 x} = \displaystyle\int \frac{\sin 2x}{\sin^3 x}dx$; $\dfrac{y}{\sin^3 x} = 2\displaystyle\int \cot x \cdot \operatorname{cosec} x\, dx = -2\operatorname{cosec} x + c$

$y = -2\sin^2 x + 4\sin^3 x \quad \left(\because y\left(\dfrac{\pi}{2}\right) = 2\right)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

Sol. $\displaystyle x\int_0^x g(t)\,dt + \int_0^x (1-t)g(t)\,dt = x^4 + x^2$

differentiate w.r.t. 'x'

$$x\,g(x) + \int_0^x g(t)\,dt + (1-x)\,g(x) = 4x^3 + 2x \quad \text{...(1)}$$

1. From (1)

$$\int_0^x g(t)\,dt + g(x) = 4x^3 + 2x$$

Let $g(x) + g'(x) = 12x^2 + 2 \Rightarrow \dfrac{dy}{dx} + y = 12x^2 + 2 \qquad (\because y = g(x))$

2. Put $x = 0$ in (1) we get $g(0) = 0$

Paragraph for Question Nos. 3 to 5

3. $f(g(x)) = e^{-2x}$

$$\frac{x \cdot [f(g(x))]'}{f(g(x))} = \frac{[g(f(x))]'}{g[f(x)]}$$

$\Rightarrow \quad g(f(x)) = e^{-x^2}$

$H(x) = e^{-(x-1)^2+1}$

4. $f(g(0)) + g(f(0)) = 2$

5. $H(x)_{\max} = e$

Paragraph for Question Nos. 6 to 8

Sol. $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{e^x g(h)}{h} + \lim_{h\to 0} g(x)\left(\frac{e^h - 1}{h}\right)$ $(\because g'(0) = 2)$

$= 2e^x + g(x)$

$\frac{dy}{dx} - y = 2e^x \Rightarrow y = 2x e^x + ce^x$

$\Rightarrow \quad y = 2x e^x \qquad (\because g(0) = 0)$

Exercise-4 : Matching Type Problems

1. (A) $y\frac{dx}{dy} - x = y^2 \frac{dx}{dy} + 1 \Rightarrow (y^2 - y)\frac{dx}{dy} = -(1+x) \Rightarrow \frac{dx}{1+x} = -\frac{dy}{y(y-1)}$

(B) $y\frac{dx}{dy} + 2x = 10y^3 \Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = 10y^2$

I.E. $= e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$

$d(xy^2) = 10y^4$

(C) $\frac{dy}{dx} = y'$

$y' y''' = (3y'')^2$

$\frac{y'''}{y''} = \frac{3y''}{y'}$ then integrate it.

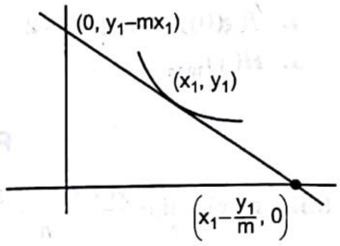
(D) Put $x^2 = t$

$\frac{dt}{dy} + \frac{t}{y} = \frac{1}{y^3}$

then solve it.

Exercise-5 : Subjective Type Problems

1. $x^a \cdot y = \lambda^a$; $\quad y = \dfrac{\lambda^a}{x^a}$



$$\frac{dy}{dx} = -a\lambda^a x^{-a-1} = -a\frac{x^a \cdot y}{x^{a+1}}$$

$$\Rightarrow \quad m = \frac{-ay_1}{x_1}$$

$$A = \frac{1}{2}|y_1 - mx_1|\left|x_1 - \frac{y_1}{m}\right| = \frac{1}{2}y_1x_1(1+a)^2$$

$$= \frac{1}{2}\lambda^a \cdot x_1^{1-a}(1+a)^2$$

For A to be constant $1 - a = 0$.

2. $\dfrac{dy}{dx} = xy(1+y)$

$$\int \frac{dy}{(1+y)\,y} = \int x\,dx$$

$$\frac{2y}{1+y} = e^{\frac{x^2}{2}} \qquad (\because f(0) = 1)$$

$$\Rightarrow \quad f(2) = \frac{e^2}{2 - e^2}$$

3. $y^2 = \cos^2 x + 2$

$$2y\frac{dy}{dx} = -\sin 2x$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\cos 2x$$

$$y^4 + y^3\frac{d^2y}{dx^2} = (\cos^2 x + 2)^2 + (\cos^2 x + 2)\left[-\left(\frac{dy}{dx}\right)^2 - \cos 2x\right] = 6$$

4. $\displaystyle \lim_{t \to x+1} \frac{t^2 f(x+1) - (x+1)^2 f(t)}{f(t) - f(x+1)} = 1$

$$\Rightarrow \quad \lim_{t \to x+1} \frac{2tf(x+1) - (x+1)^2 f'(t)}{f'(t)} = 1$$

$$\Rightarrow \quad [x+1][2f(x+1) - (x+1)f'(x+1)] = f'(x+1)$$

$\Rightarrow \quad f'(x) = \dfrac{2xf(x)}{x^2+1}$

$\Rightarrow \quad f(x) = x^2 + 1$

$\Rightarrow \quad \lim\limits_{x \to 1} \dfrac{\ln(f(x)) - \ln 2}{x-1} = \lim\limits_{x \to 1} \dfrac{f'(x)}{f(x)} = 1$

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Chapter 8 – Quadratic Equations

8 QUADRATIC EQUATIONS

Exercise-1 : Single Choice Problems

1. Let $3^{x/2} = a, 2^y = b$

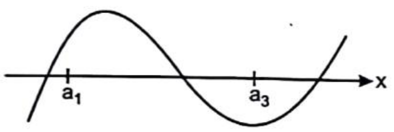
$a^2 - b^2 = 77, a - b = 7 \Rightarrow a = 3^{x/2} = 9 \Rightarrow x = 4$

$b = 2^y = 2 \Rightarrow y = 1$

2. $f(x) = \prod_{i=1}^{3}(x - a_i) + \sum_{i=1}^{3} a_i - 3x = (x - a_1)(x - a_2)(x - a_3) + (a_1 + a_2 + a_3) - 3x$

$f(a_1) = a_2 + a_3 - 2a_1 > 0 \qquad (a_1 < a_2 < a_3)$

$f(a_3) = a_1 + a_2 - 2a_3 < 0$



3. $x^4 - 2ax^2 + x + a^2 - a = 0$

$a^2 - a(2x^2 + 1) + x^4 + x = 0$

$$a = \frac{2x^2 + 1 \pm (2x - 1)}{2}$$

$a = x^2 + x, \quad a = x^2 - x + 1$

$a \geq -\frac{1}{4}, \qquad a \geq \frac{3}{4} \qquad (\because x \in R)$

4. $x^3 - 3x^2 - 4x + 12 = 0$ (roots α, β, γ)

Equation whose roots are $\alpha - 3, \beta - 3, \gamma - 3$ is

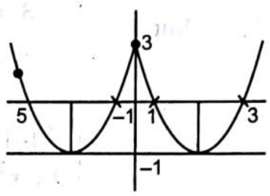
$(x + 3)^3 - 3(x + 3)^2 - 4(x + 3) + 12 = 0$

$$f(x) = x^3 + 6x^2 + 5x = 0 \quad (\text{roots } \alpha - 3, \beta - 3, \gamma - 3)$$

5. $|K-1|<3$

$-3<K-1<3$

$-2<K<4$



6. $\dfrac{x}{x+6}-\dfrac{1}{x}\le 0 \Rightarrow \dfrac{x^2-x-6}{x(x+6)}\le 0 \Rightarrow \dfrac{(x-3)(x+2)}{x(x+6)}\le 0$

+ –6 – –2 + 0 – 3 +

$x\in(-6,-2]\cup(0,3]$

7. $P(x)=x^4-8x^2+15+2x^3-6x=(x^2-3)(x^2-5)+2x(x^2-3)$

$=(x^2-3)(x^2+2x-5)$

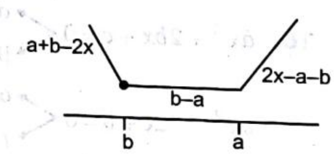
$Q(x)=(x+2)(x^2+2x-5)$

8. $a=1, h=\dfrac{\lambda}{2}, b=1, g=\dfrac{-5}{2}, f=\dfrac{-7}{2}, c=6$

$$\begin{vmatrix} 1 & \lambda/2 & -5/2 \\ \lambda/2 & 1 & -7/2 \\ -5/2 & -7/2 & 6 \end{vmatrix}=0 \Rightarrow \lambda=\frac{5}{2},\frac{10}{3}$$

10. $\therefore$ Let $f(x)=|x-a|+|x-b|$

Suppose $a>b$



$\therefore$ $f(0)=f(1)=f(-1)$

$f(x)=$ const. in $[b,a]$

So, $b\le -1<a\le 1$

$a-b\le 2$

$\therefore$ Minimum $|a-b|=2$

12. $y=\dfrac{x^2+2x+c}{x^2+4x+3c} \Rightarrow (y-1)x^2+2(2y-1)x+(3cy-c)=0$ $(D\ge 0)$

$D\ge 0\ \forall\ y\in R$ and $D\le 0$

But at $c=0$ and 1 there will be common factors among numerator and denominator.

$\Rightarrow$ $c(c-1)<0$

13. $f(t)=t^2-mt+2=0$

$f(2)<0$

$\Rightarrow$ $4-2m+2<0 \Rightarrow m>3$

But $\frac{3|x|}{9+|x|^2} = \frac{3}{\frac{9}{|x|}+|x|} \le \frac{3}{6}$ (by A.M. G.M. in equality)

$$\left(\frac{3|x|}{9+x^2}\right)^m \le \frac{1}{2^m} < 1 \quad [\because m > 3]$$

So, $\left[\left(\frac{3|x|}{9+x^2}\right)^m\right] = 0$

14. $x^2(x^6 - 24x^5 - 18x^3 + 39) = -3 \times 5 \times 7 \times 11$

If 'x' is integer, then there is no value of 'x'.

15. $m^4 + \frac{1}{m^4} = 119$

$\Rightarrow m^2 + \frac{1}{m^2} = 11$

$\Rightarrow \left(m - \frac{1}{m}\right)^2 = 9$

$$\left|m^3 - \frac{1}{m^3}\right| = \left|\left(m - \frac{1}{m}\right)\left(m^2 + \frac{1}{m^2} + 1\right)\right| = |3 \times 12| = 36$$

16. $ax^2 + 2bx + c = 0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix}$

$ax^2 + 2cx + b = 0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \gamma\end{smallmatrix}$

By condition of common root

$\Rightarrow$ Common root $\alpha = \frac{1}{2}$ and $\frac{a}{4} + b + c = 0$

$\Rightarrow \beta = \frac{2c}{a}$ and $\gamma = \frac{2b}{a}$

Equation whose roots are β and γ is

$$x^2 - \left(\frac{2c}{a} + \frac{2b}{a}\right)x + \frac{4bc}{a^2} = 0$$

$$2a^2x^2 + a^2x + 8bc = 0 \quad \left(\frac{a}{4} + b + c = 0\right)$$

17. $9x^2(x-1) - 1(x-1) = 0$

$$x = 1, x = \frac{1}{3}, -\frac{1}{3}$$

$$\cos\alpha = 1, \cos\beta = \frac{1}{3}, \cos\gamma = -\frac{1}{3}$$

$$\alpha = 0, \beta + \gamma = \pi$$

$\therefore$ $$(\Sigma\alpha, \Sigma\cos\alpha) = (\pi, 1) = \text{centre}$$

$$\left[2\sin^{-1}\left(\tan\frac{\pi}{4}\right), 4\right] = \left[2\left(\frac{\pi}{2}\right), 4\right] = (\pi, 4) \rightarrow \text{point lies on the circle.}$$

$\therefore$ Radius is 3.

18. $y = \dfrac{11x^2 - 12x - 6}{x^2 + 4x + 2}$

$(y - 11)x^2 + (4y + 12)x + (2y + 6) = 0 \ \forall \ x \in R$

$$D \geq 0$$

$(4y + 12)^2 - 4(y - 11)(2y + 6) \geq 0$

$$y^2 + 20y + 51 \geq 0$$

$$(y + 17)(y + 3) \geq 0$$

$$y \in (-\infty, -17] \cup [-3, \infty)$$

19. $$\frac{x + 3}{x^2 - x - 2} - \frac{1}{x - 4} \geq 0$$

$$\frac{(x^2 - x - 12) - (x^2 - x - 2)}{(x^2 - x - 2)(x - 4)} \geq 0$$

$$\frac{-10}{(x - 2)(x + 1)(x - 4)} \geq 0$$

$\Rightarrow$ $(x + 1)(x - 2)(x - 4) < 0$

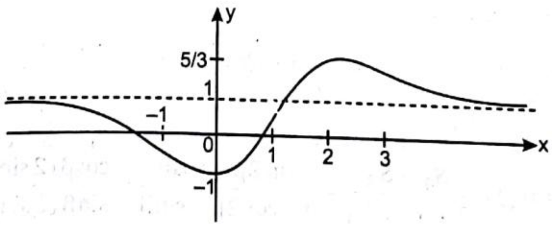
$$x \in (-\infty, -1) \cup (2, 4)$$

20. $x = 4 + 3i$

$(x - 4)^2 = -9 \ \Rightarrow \ x^2 - 8x + 25 = 0$

$x^3 - 4x^2 - 7x + 12 = (x^2 - 8x + 25)(x + 4) - 88 = -88$

21. $f(x) = \dfrac{x^2 + x - 1}{x^2 - x + 1}$



By graph : Min. = $f(0)$; Max. = $f(2)$

If $x \in [-1, 3]$; $y_{max.} = \dfrac{5}{3}$

22. By graph min. = $f(0)$; max. = $f(1)$
if $x \in [-1, 1]$; $y \in [-1, 1]$

23.
$$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$$

$x^2 + x(p+q-2r) + pq - r(p+q) = 0$

If one root is α. Then other root must be $-\alpha$.

$$p + q - 2r = 0 \Rightarrow r = \frac{p+q}{2}$$

Product of the roots $= pq - r(p+q) = pq - \frac{(p+q)^2}{2} = -\frac{(p^2+q^2)}{2}$

24. If $a_1x^2 + b_1x + c_1 = 0$ has one root α.

$\Rightarrow \quad a_2x^2 + b_2x + c_2 = 0$ has one root $\frac{1}{\alpha}$.

$\Rightarrow \quad c_2x^2 + b_2x + a_2 = 0$ has one root α

Condition of common root is

$$(a_1a_2 - c_1c_2)^2 = (a_1b_2 - b_1c_2)(a_2b_1 - b_2c_1)$$

25. If $\alpha^2 - 5\alpha + 3 = 0$ and $\beta^2 - 5\beta + 3 = 0$

$\Rightarrow \quad x^2 - 5x + 3 = 0$ has two roots α and β.

$\Rightarrow \quad \alpha + \beta = 5, \; \alpha\beta = 3$

Sum of the roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{19}{3}$

Product of roots $= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

Equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is $3x^2 - 19x + 3 = 0$

26. $|\alpha - \beta| = |\alpha_1 - \beta_1|$

$a^2 - 4b = b^2 - 4a$

$a^2 - b^2 = 4(b - a)$

$(a - b)(a + b + 4) = 0$

$a \neq b \Rightarrow a + b + 4 = 0$

27. $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{S_1 - S_3}{1 - S_2 + S_4} = \frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \frac{\cos\beta\,(2\sin\beta - 1)}{\sin\beta\,(2\sin\beta - 1)} = \cot\beta$

28. $(a^2 + b^2)x^2 + 2x(bd + ac) + (c^2 + d^2) = 0$

$(a^2x^2 + 2acx + c^2) + (b^2x^2 + 2bdx + d^2) = 0$

$$(ax+c)^2 + (bx+d)^2 > 0$$

$\Rightarrow$ This equation has imaginary roots.

29. If α, β are roots of $ax^2 + bx + c = 0$

$\Rightarrow$ $\alpha + 2, \beta + 2$ are roots of $a(x-2)^2 + b(x-2) + c = 0$

$\Rightarrow$ $ax^2 + x(b - 4a) + 4a - 2b + c = 0$

30.
$$\alpha + \beta = 1 + \lambda$$
$$\alpha\beta = \lambda - 2$$
$$\alpha + \beta - \alpha\beta = 3$$
$$(\alpha - 1)(\beta - 1) = -2$$

$\Rightarrow$ atleast one root is positive.

31. $D \geq 0 \Rightarrow 3k^2 + 8k - 16 \leq 0 \Rightarrow -4 \leq k \leq \frac{4}{3}$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = k^2 - 2(k^2 + 2k - 4) = -k^2 - 4k + 8 = 12 - (k+2)^2$$

32.
$$P(x) = (x-2)\,Q_1(x) + R(x)$$
$$Q(x) = (x-2)\,Q_2(x) + R(x)$$

$\Rightarrow$ $P(2) = Q(2)$

33. $a + b = -a$ and $ab = b$

if $b \neq 0$, $a = 1$ and $b = -2$

$$x^2 + ax + b = x^2 + x - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$$

34. $$x^2 + \left(\frac{b}{a}\right)\left(\frac{c}{a}\right)x + \left(\frac{c}{a}\right)^3 = 0$$

$$x^2 - (\alpha + \beta)\cdot\alpha\beta\, x + \alpha^3\beta^3 = 0 \begin{cases} \alpha^2\beta \\ \alpha\beta^2 \end{cases}$$

35. $x^2 + 2(a + b + c)\,x + 6k(ab + bc + ca) = 0$

$D \geq 0$

$\Rightarrow$ $4(a+b+c)^2 - 24k(ab + bc + ca) \geq 0$

$\Rightarrow$ $k \leq \frac{1}{6}\left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} + 2\right)$

also, $|a - b| < c$, $|b - c| < a$, $|c - a| < b$

$\Rightarrow$ $a^2 + b^2 + c^2 - 2(ab + bc + ca) < 0$

$\Rightarrow$ $\frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2 \Rightarrow k < \frac{2}{3}$

36. $9|x|^2-18|x|+5=0$

$\Rightarrow\quad (3|x|-1)(3|x|-5)=0$

$\Rightarrow\quad x=\pm\frac{1}{3},\pm\frac{5}{3}$

and $x^2-x-2>0\ \Rightarrow (x-2)(x+1)>0\ \Rightarrow x<-1$ or $x>2$

37. Difference of roots is same in both equation

$$b^2-c=B^2-C$$

38. $|x-p|+|x-15|+|x-p-15|=(x-p)-(x-15)-(x-P-15)=30-x$

min. $=15$

39. $4p(q-r)x^2-2q(r-p)x+r(p-q)=0$ → $\alpha=-1/2$, $\beta=-1/2$

If $x=-\frac{1}{2}$ is also the root of $4x^2-2x-m=0$

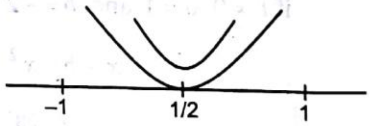
$\Rightarrow\quad m=2$

40. Let $\cos x=t$

$\Rightarrow\quad t\in[-1,1]$

$\Rightarrow\quad kt^2-kt+1\geq 0\ \forall\ t\in[-1,1]$

Case I : $k\geq 0$



x coordinate of vertex is $\frac{1}{2}$.

$\Rightarrow\quad f\left(\frac{1}{2}\right)\geq 0$

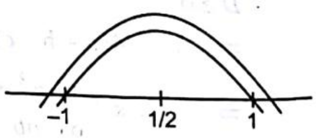
$\Rightarrow\quad \frac{k}{4}-\frac{k}{2}+1\geq 0$

$\Rightarrow\quad k\leq 4$

Also, $\quad k\geq 0$

$\Rightarrow\quad k\in[0,4]$

Case II : $k<0$



$\Rightarrow\quad f(1)\geq 0$ and $f(-1)\geq 0$

$k-k+1\geq 0$ and $k+k+1\geq 0$

$\Rightarrow\quad 1\geq 0$ and $k\geq -\frac{1}{2}$

Also, $\quad k<0$

$\Rightarrow\quad k\in\left[-\frac{1}{2},0\right)\quad\Rightarrow\quad k\in\left[-\frac{1}{2},4\right]$

41. $\frac{1+x}{1-x}=y \Rightarrow x=\frac{y-1}{y+1}$

$H(y)=3\left(\frac{y-1}{y+1}\right)^3-2\left(\frac{y-1}{y+1}\right)+5=0$

$H(y)=3(y-1)^3-2(y-1)(y+1)^2+5(y+1)^3=0$

$H(y)=3(y^3-3y^2+3y-1)-2(y-1)(y^2+2y+1)+5(y^3+3y^2+3y+1)=0$

$H(y)=3(y^3-3y^2+3y-1)-2(y^3+y^2-y-1)+5(y^3+3y^2+3y+1)=0$

$H(y)=3y^3+2y^2+13y+2=0$

$H'(x)=9x^2+4x+13 \Rightarrow D<0$

$H(x)>0\ \forall\ x>0$

Hence, it has one –ve real root.

42. $(\lambda^2+\lambda-2)x^2+(\lambda+2)x-1<0\ \forall\ x\in R$

$\lambda^2+\lambda-2<0 \cap (\lambda+2)^2+4(\lambda^2+\lambda-2)<0$

$(\lambda+2)(\lambda-1)<0 \cap 5\lambda^2+8\lambda-4<0$

$\lambda\in(-2,1)\cap\lambda\in\left(-2,\frac{2}{5}\right)$

$\Rightarrow\ \lambda\in\left(-2,\frac{2}{5}\right)$

$\lambda=-2$ is also the solution of this equation.

43. $\alpha=1,\beta=1,\gamma=1,\delta=1$ (as) $(\alpha-1)^2+(\beta-1)^2+(\gamma-1)^2+(\delta-1)^2=0$

∴ The roots of given equation is equal to 1.

∴ $S_2=\frac{a_2}{a_0}=6$

44. $|x-1|+|x-2|+|x-3|\geq 6$

Case I : $x\geq 3$

$3x-6\geq 6 \Rightarrow x\geq 4$

Case II : $2<x<3$

$x\geq 6$ (Not possible)

Case III : $1\leq x\leq 2$

$4-x\geq 6$

$\Rightarrow$ $x\leq -2$ (Not possible)

Case IV : $x<1$

$6-3x\geq 6$

$x\leq 0$

$x\in(-\infty,0]\cup[4,\infty)$

45. 

Case-I : $f(0) > 0 \cap f(3) \le 0$ **Case-II :** $f(3) > 0 \cap f(0) \le 0$

46. $x^3 + 3px^2 + 3qx + r = 0 \begin{matrix} \nearrow \alpha \\ \to \beta \\ \searrow \gamma \end{matrix}$

$$\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma} \ldots\ldots \qquad (\because\ \alpha, \beta, \gamma \text{ are in H.P.})$$

$$\Rightarrow \quad \frac{3}{\beta} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$

$$\Rightarrow \quad \beta = -\frac{r}{q} \qquad \text{which satisfy the given equation.}$$

47. $4y^2 + 4xy + (x+6) = 0\ \forall\ y \in R$

$D \ge 0\ \Rightarrow\ x^2 - x - 6 \ge 0$

48. $\log_{\cos x^2}(3-2x) < \log_{\cos x^2}(2x-1)$

$0 < \cos x^2 < 1 \cap 3 - 2x > 2x - 1 \cap 3 - 2x > 0 \cap 2x - 1 > 0$

$x < 1 \qquad x < 3/2 \qquad x > 1/2$

49. $px^2 + qx + r = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$\Rightarrow \quad \alpha\beta < 0$

$\alpha(x-\beta)^2 + \beta(x-\alpha)^2 = (\alpha+\beta)x^2 - 4\alpha\beta x + \alpha\beta(\alpha+\beta) = 0$

Product of roots $= \alpha\beta < 0$

$D = 16\alpha^2\beta^2 - 4\alpha\beta(\alpha+\beta)^2 = -4\alpha\beta(\alpha-\beta)^2 > 0$

50. $x^3 + 2x^2 - 4x - 4 = 0 \begin{matrix} \nearrow a \\ \to b \\ \searrow c \end{matrix}$

$4x^3 + 4x^2 - 2x - 1 = 0 \begin{matrix} \nearrow 1/a \\ \to 1/b \\ \searrow 1/c \end{matrix}$

$q = 1, r = -\dfrac{1}{2}, s = -\dfrac{1}{4}$

51. $\log_2(x^2 + 3x) \le 2$

$0 < x^2 + 3x \le 4$

52. $k - 2 > 0 \cap D < 0$

$k > 2 \cap (k+6)(k-4) > 0$

$\Rightarrow \quad k > 4$

53. $\alpha\beta < 0$

$\dfrac{3m-8}{m-2} < 0 \;\Rightarrow\; 2 < m < \dfrac{8}{3}$

54. $\log_6\left(\dfrac{x^2+x}{x+4}\right) > 1$

$\dfrac{x^2+x}{x+4} > 6 \;\Rightarrow\; \dfrac{x^2-5x-24}{x+4} > 0 \;\Rightarrow\; \dfrac{(x-8)(x+3)}{x+4} > 0$

55. $ax^2 + c = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$\alpha + \beta = 0,\ \alpha\beta = \dfrac{c}{a}$

$\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 + \beta^2 - \alpha\beta) = 0$

56. $(k-1)x^2 - (k+1)x + (k+1) > 0 \;\forall\; x \in R$

$k - 1 > 0 \;\cap\; (k+1)^2 - 4(k-1)(k+1) < 0$

$k > 1 \quad \cap \; (k+1)(3k-5) > 0$

$\Rightarrow \quad k > \dfrac{5}{3}$

57. $y = -2x^2 - 4ax + k$; abscissa corresponding to the vertex is $-\dfrac{b}{2a}$ *i.e.*, $\left(\dfrac{4a}{-4}\right) = -2 \Rightarrow a = 2$

now, $\quad y(-2) = 7$

$7 = -8 + 16 + k \Rightarrow k = -1$

58. If $a + b + c = 0$

Sum of coefficient $(b + c - a) + (c + a - b) + (a + b - c) = a + b + c = 0$

$\Rightarrow$ $x = 1$ is one root of the equation.

$\Rightarrow$ other root $= \dfrac{a+b-c}{b+c-a}$

59. $x^3 - ax^2 + bx - c = 0 \begin{matrix} \nearrow \alpha \\ \rightarrow -\alpha \\ \searrow \beta \end{matrix}$

Sum of roots $\alpha - \alpha + \beta = a \Rightarrow \beta = a$

If β is root of the equation, then $ab = c$.

60. $\alpha'\beta' = 2q^2 - r = 2\alpha^2\beta^2 - (\alpha^4 + \beta^4) = -(\alpha^2 - \beta^2)^2 < 0$

62. In ΔABC,

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$$

If $\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}$ are in A.P. then $\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$

$\Rightarrow$ $$\cot\frac{A}{2}\cot\frac{C}{2}=3$$

63. $f(x)=x\ \forall\ x\in[-9,9]$

64. $$(3|x|-3)^2=|x|+7$$

$\Rightarrow\ (|x|-2)(9|x|-1)=0$

$$|x|=2,\frac{1}{9}\Rightarrow x=\pm 2,\pm\frac{1}{9}$$

$$y=\sqrt{x(x-4)}$$

$D_f:(-\infty,0]\cup[4,\infty)$

65. $x^2+3|x|+2=0\Rightarrow(|x|+2)(|x|+1)=0$

66. $x^2-bx+c=0$ with roots α and $\alpha+1$

Sum of roots $2\alpha=b-1\Rightarrow\alpha=\frac{b-1}{2}$

If α is the root of equation, then $\left(\frac{b-1}{2}\right)^2-b\left(\frac{b-1}{2}\right)+c=0\Rightarrow b^2-4c=1$

67. $y=\dfrac{3x^2+9x+17}{3x^2+9x+7}$

$$3(y-1)x^2+9(y-1)x+(7y-17)=0$$

$y-1\neq 0$ then $D\geq 0$

$$81(y-1)^2-12(y-1)(7y-17)\geq 0$$

$$(y-1)(y-41)\leq 0$$

$$1<y\leq 41$$

68. $\dfrac{x^2+2x+7}{2x+3}-6<0\ \forall\ x\in R$

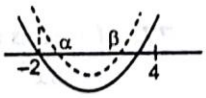
$\dfrac{x^2-10x-11}{(2x+3)}<0$

$\dfrac{(x-11)(x+1)}{2x+3}<0$

$x\in\left(-\infty,-\frac{3}{2}\right)\cup(-1,11)$

69. $y = \dfrac{3x-2}{7x+5} \Rightarrow x = \dfrac{5y+2}{3-7y} \Rightarrow y \in R - \left\{\dfrac{3}{7}\right\}$

70. $\dfrac{x+2}{x-4} \le 0 \Rightarrow x \in [-2, 4)$



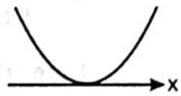
$x^2 - ax - 4 \le 0$

$f(-2) \ge 0 \cap f(4) > 0$

$a \ge 0 \cap a < 3$

$\Rightarrow \quad a \in [0, 3)$

71. $P(x) = (P-3)x^2 - 2Px + (3P-6) \; \forall \; x \in R$



$P - 3 > 0 \quad \cap \quad D = 0$

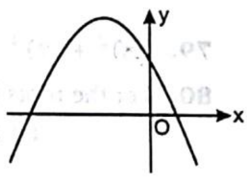
$P > 3 \quad \cap \quad P^2 - (P-3)(3P-6) = 0$

$\Rightarrow \quad P = 6$

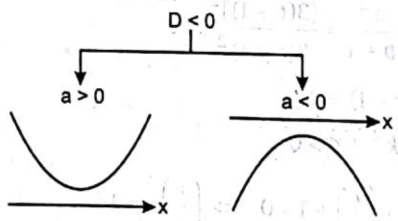
72. Graph is downward $\Rightarrow a < 0$

Graph cut y-axis $\Rightarrow c > 0$

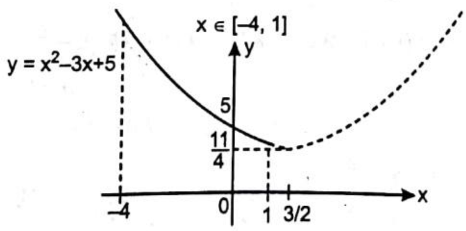
x-coordinate of vertex $\dfrac{-b}{2a} < 0 \Rightarrow b < 0$



73. $ax^2 + bx + c = 0$ does not have real roots.



74.



$y \in [3, 33]$

75. $3x^2 - 17x + 10 = 0 \Rightarrow (x-5)(3x-2) = 0$

If $x = 5$ is common root, then $m = 0$

If $x = \dfrac{2}{3}$ is common root, then $m = \dfrac{26}{9}$

76. $x^2+(y+2)x-(y^2+y-1)=0$

$D\geq 0 \Rightarrow (y+2)^2+4(y^2+y-1)\geq 0 \Rightarrow y\in\left(-\infty,-\frac{8}{5}\right]\cup[0,\infty)$

77. If $x=3$ is root of this equation, then $k=-5$

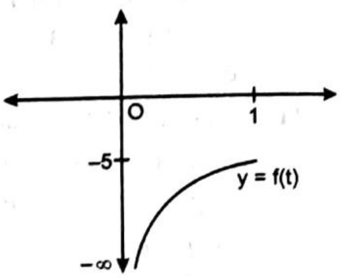
$\Rightarrow\quad 3x^4-6x^3-5x^2-8x-12=(x-3)(3x^2+4)(x+1)$

78. $a=-\dfrac{(4+\sin^4 x)}{\sin^2 x}$ put $\sin^2 x=t \Rightarrow t\in[0,1]$

$a=-\left(\dfrac{4}{t}+t\right)=f(t)$

Here, $f'(t)=\dfrac{4}{t^2}-1>0$

$\therefore$ For atleast one real root, $a\in(-\infty,-5]$



79. $(rs)^2+(st)^2+(tr)^2=(rs+st+tr)^2-2rst(r+s+t)=b^2-2(-c)(-a)$

80. Let the roots be $t, t+1$ and $t+2$.

$$t+(t+1)+(t+2)=-a \quad\Rightarrow 3(t+1)=-a$$

$$\sum t(t+1)=b \quad\Rightarrow\quad b+1=3(t+1)^2$$

$$\frac{a^2}{b+1}=\frac{[3(t+1)]^2}{3(t+1)^2}=3$$

81. $(3x^2+kx+3)(x^2+kx-1)=0$

$D_1=k^2-36$ and $D_2=k^2+4>0$

82. $\dfrac{1}{r+s}=\dfrac{1}{r}+\dfrac{1}{s} \Rightarrow \left(\dfrac{r}{s}\right)^2+\left(\dfrac{r}{s}\right)+1=0 \Rightarrow \left(\dfrac{r}{s}\right)^3=1$

84. If $x\in(-\infty,-2]\cup[3,\infty)$

$x^2-2x-8=0 \Rightarrow x=-2,4$

if $x\in(-2,3)$

$x^2=4 \Rightarrow x=\pm 2$

85. $5x^2+12x+3=0$ has $D<0$

$\Rightarrow$ Both roots common.

86.

$$\alpha+\beta+\gamma=6$$
$$\alpha\beta+\beta\gamma+\alpha\gamma=5$$
$$\alpha\beta\gamma=1$$

$\Rightarrow\quad \alpha^2+\beta^2+\gamma^2=26$

$\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2=13$

87. $2x^2 - 6x + k = 0 \left\langle \begin{matrix} \frac{\alpha + 5i}{2} \\ \frac{\alpha - 5i}{2} \end{matrix} \right.$

Sum of roots $= \alpha = 3$

Product of roots $= \dfrac{\alpha^2 + 25}{4} = \dfrac{k}{2} \Rightarrow k = 17$

88. $x_1^2 + x_2^2 = (k-2)^2 - 2(k^2 + 3k + 5) = -(k^2 + 10k + 6) \le 18$

89. $a(x^2 - x + 1) - (x^2 + x + 1) \ge 0$

$\Rightarrow \qquad a \ge \dfrac{x^2 + x + 1}{x^2 - x + 1}$

90. $f(1) = \lambda - 13 > 0 \quad \Rightarrow \lambda > 13$

$f(2) = \lambda - 18 < 0 \quad \Rightarrow \lambda < 18$

$f(3) = \lambda - 15 > 0 \quad \Rightarrow \lambda > 15$

$\Rightarrow \lambda \in (15, 18)$

94. $D = (b-c)^2 + 4a(2b + a + c) = (b-c)^2 + (4ac - 4b^2) + (2a + 2b)^2 > 0$

95. $x^3 - x + 1 = 0 \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right.$

$(1-x)^3 - x^2(1-x) + x^3 = 0 \left\langle \begin{matrix} \frac{1}{a+1} \\ \frac{1}{b+1} \\ \frac{1}{c+1} \end{matrix} \right.$

$\Rightarrow \qquad x^3 + 2x^2 - 3x + 1 = 0$

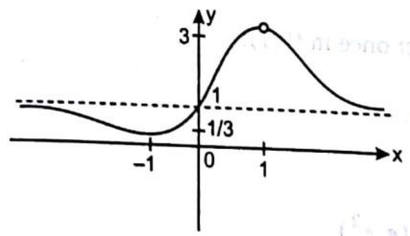
Sum of roots $= \dfrac{1}{1+a} + \dfrac{1}{1+b} + \dfrac{1}{1+c} = -2$

96. $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 \ge 0 \; \forall \; x \in R$

$D \le 0$

$\Rightarrow \quad k^2 - 6k - 8 \le 0 \quad \Rightarrow 2 \le k \le 4$

97. $f(x) = \dfrac{x^3 - 1}{(x-1)(x^2 - x + 1)} = \dfrac{x^2 + x + 1}{x^2 - x + 1} \qquad (\because x \ne 1)$



98. $\frac{2x^2+2}{x^2+mx+4} > 0 \ \forall\ x \in R \quad \Rightarrow x^2+mx+4>0 \ \forall\ x \in R$

$\Rightarrow D<0 \Rightarrow m^2-16<0$

99. $x^2-2|a+1|x+1=0$

$D \geq 0 \Rightarrow 4(a+1)^2-4 \geq 0 \Rightarrow a \in (-\infty,-2] \cup [0,\infty)$

100. $P(x)=a_1x^2+2b_1x+c_1>0;\ D_1=4(b_1^2-a_1c_1)<0,\ a_1>0, c_1>0$

$Q(x)=a_2x^2+2b_2x+c_2>0;\ D_2=4(b_2^2-a_2c_2)<0,\ a_2>0, c_2>0$

$f(x)=a_1a_2x^2+b_1b_2x+c_1c_2$

$D=b_1^2b_2^2-4a_1a_2c_1c_2<0$

101. $x^2-2x+4=-3\cos(ax+b)$

$(x-1)^2+3=-3\cos(ax+b)$

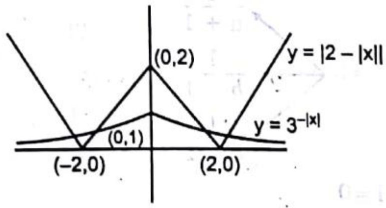
$\Rightarrow \quad x=1$ and $ax+b=\pi$

102. $\alpha+\beta=\alpha+\alpha\cdot r=4$

$\gamma+\delta=\alpha\cdot r^2+\alpha r^3=36 \quad \Rightarrow r=3, \alpha=1$

$A=3,\ B=243=3^5$

103.



104. We have $4x^2-16x+15<0 \Rightarrow \frac{3}{2}<x<\frac{5}{2} \Rightarrow \cot\alpha=2$, the integral solution of the given inequality and $\sin\beta=\tan 45°=1$

$\therefore \quad \sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\frac{1}{1+\cot^2\alpha}-1=\frac{1}{1+4}-1=-\frac{4}{5}$

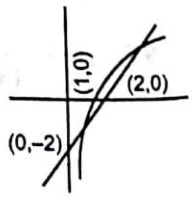
105. $f_1(x)=f_2(x)$

$\Rightarrow \quad 2+\log_e x=x$

$\Rightarrow \quad \log_e x=(x-2)$

Clearly graphs intersect once in (0,1).

Now check



$\Rightarrow \quad g(x)=2+\ln x-x$

$g(e)>0$

$g(e^2)<0$

$\Rightarrow \quad$ one root between (e, e^2)

106. $x^4-3x^3-2x^2-3x+1=0 \qquad \Rightarrow x^2-3x-2-\frac{3}{x}+\frac{1}{x^2}=0$

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-2=0 \Rightarrow \left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)-4=0$

$\Rightarrow t^2-3t-4=0$ (where $x+\frac{1}{x}=t$)

$\Rightarrow (t-4)(t+1)=0 \Rightarrow t=4$ or $t=-1$

$\Rightarrow x+\frac{1}{x}=4$ or $x+\frac{1}{x}=-1$

Real solutions are from $x+\frac{1}{x}=4 \Rightarrow x^2+1=4x \Rightarrow x^2-4x+1=0$

Hence, sum of roots = 4.

107. $f(x)=x^2-(k+4)+k^2-12$

$f(4)=16-4(k+4)+k^2-12<0$

$\Rightarrow -2<k<6$

108. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=k^2-2(k^2+2k-4)=-k^2-4k+8$

Maximum value = 12

109. $f(x)=a^x-x\ln a$

$f'(x)=(a^x-1)\cdot\ln a$

110. As a, b and c are the roots of $x^3+2x^2+1=0$, we have

$$a+b+c=-2$$
$$ab+bc+ca=0$$

Now, for finding the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, evaluating using first row, we get

$$\begin{aligned} a(bc-a^2)-b(b^2-ac)+c(ab-c^2) &= abc-a^3-b^3+abc+abc-c^3 \\ &= 3abc-a^3-b^3-c^3 \\ &= -(a^3+b^3+c^3-3abc) \\ &= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= -(-2)[(-2)^2-3(0)]=8 \end{aligned}$$

111. $x^2+px+q=0,\ p,q\in R, q\neq 0 \quad \alpha,\beta$ real roots.

$g(x)=0 \qquad \alpha+\frac{1}{\alpha}, \beta+\frac{1}{\beta}$

$$\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}=\left(\alpha+\frac{1}{\alpha}\right)\left(\beta+\frac{1}{\beta}\right)$$

$$-p+\frac{-p}{q}=\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}$$

$$-p+\frac{-p}{q}=q+\frac{p^2-2q}{q}+\frac{1}{q}$$

$$-pq-p=q^2+p^2-2q+1$$

$$p^2+p(p+1)+q^2-2q+1=0$$

$$(q+1)^2-4(q^2-2q+1)\geq 0$$

$$q^2+2q+1-4q^2+8q-4\geq 0$$

$$-3q^2+10q-3\geq 0$$

$$3q^2-2q-q+3\leq 0$$

$$3q(q-3)-(q-3)\leq 0$$

$$\left[\frac{1}{3},3\right]$$

112. $\ln(x^2+5x)=\ln(x+a+3) \Rightarrow x^2+5x=x+a+3>0$

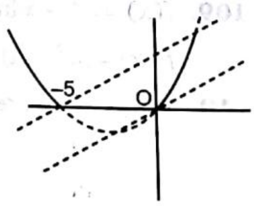
$$a+3>0$$

$$a>-3$$

$y=x+a+3 \Rightarrow -5+a+3\leq 0$

$$a\leq 2$$

$$-3<a\leq 2$$



113. $f(x)=x^2+\dfrac{1}{x^2}-6x-\dfrac{6}{x}+2=\left(x+\dfrac{1}{x}\right)^2-6\left(x+\dfrac{1}{x}\right)$

Let $x+\dfrac{1}{x}=t$

$f(x)=t^2-6t\ \forall\ t\in(-\infty,-2]\cup[2,\infty)$

min. value $=-9$ at $t=3$

114. $x^3+2x^2+2x+c=(x^2+bx+b)\left(x+\dfrac{c}{b}\right)$

$\Rightarrow\ b+\dfrac{c}{b}=2$ and $b+c=2 \Rightarrow b=c=1$

115. $\alpha\beta+\beta\gamma+\alpha\gamma=0 \Rightarrow (\alpha\beta)^3+(\beta\gamma)^3+(\alpha\gamma)^3=3(\alpha\beta)(\alpha\gamma)(\beta\gamma)$

118. $\sum_{r=1}^{\infty}(\alpha^r+\beta^r)=(\alpha+\alpha^2+\alpha^3+\ldots)+(\beta+\beta^2+\beta^3+\ldots)$

$$=\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}$$

$4x^2+2x-1=0$ with roots α, β

$4\left(\frac{x}{1+x}\right)^2+2\left(\frac{x}{1+x}\right)-1=0 \Rightarrow 5x^2-1=0$ with roots $\frac{\alpha}{1-\alpha}$, $\frac{\beta}{1-\beta}$

119. $\left(\frac{2011}{2014}\right)^x+\left(\frac{2012}{2014}\right)^x+\left(\frac{2013}{2014}\right)^x=1$

Let $f(x)=\left(\frac{2011}{2014}\right)^x+\left(\frac{2012}{2014}\right)^x+\left(\frac{2013}{2014}\right)^x \Rightarrow f(x)$ is a decreasing function for $x\in R$.

123.

$x^2+ax+12=0$ ...(1)

$x^2+bx+15=0$ ...(2) Common roots

$x^2+(a+b)x+36=0$ ...(3)

(1) + (2) − (3)

$\alpha^2=9 \Rightarrow \alpha=\pm 3$

positive root $\alpha=3$

124. $e^{\sin x}=t$

$t^2-4t-1=0 \Rightarrow t=2\pm\sqrt{5} \Rightarrow e^{\sin x}=2\pm\sqrt{5}$ (Not possible)

125. Maximum value of $f(x)=3$

Minimum value of $f(x)=-1$

126. $f(1)=\lambda-2<0$

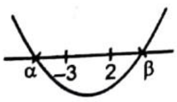
127. $2x^2+5x+7=0$ has non-real roots $\Rightarrow \frac{a}{2}=\frac{b}{5}=\frac{c}{7}$

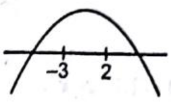
Min. value of $a+b+c=2+5+7=14$

Max. value of $a+b+c=28+70+98=196$

128. Distance $=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}=\sqrt{(1-2t)^2+t^2}=\sqrt{5t^2-4t+1}$

Min. distance $=\frac{1}{\sqrt{5}}$ at $t=\frac{2}{5}$

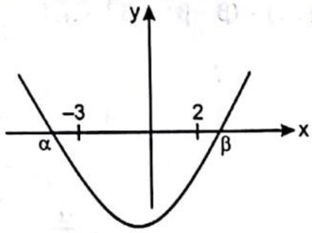
129. 



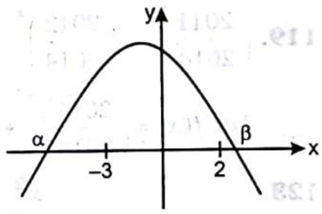
$af(1)<0$ and $af(-1)<0$; $f(-3)f(2)>0$

We have the equation $ax^2+bx+c=0$ has two roots α and β such that $\alpha<-3$ and $\beta>2$.

If $a > 0$, then we have the following graphical representation :



Then, for all $x \in [-3, 2]$, $f(x) < 0$, we have the following graphical representation :



This implies that

$f(-1) < 0$ and $f(1) < 0$

$\Rightarrow \quad a - b + c < 0$ and $a + b + c < 0$

$\Rightarrow \quad a(a + |b| + c) < 0$

If $a < 0$, then for all $x \in [-3, 2]$, $f(x) > 0$. This imply that

$\Rightarrow \quad f(-1) > 0$ and $f(1) > 0$

$\Rightarrow \quad a - b + c > 0$ and $a + b + c > 0$

$\Rightarrow \quad a(a + |b| + c) < 0$

130. Let $x^2 + 5x = t$

$$t^2 - 2t - 24 = 0 = (t - 6)(t + 4)$$

$$x^2 + 5x - 6 = 0 = (x + 6)(x - 1)$$

$$x^2 + 5x + 4 = 0 = (x + 4)(x + 1)$$

131. Case-1: $x \geq 2$

$3(x - 2) - (1 - 5x) + 4(3x + 1) = 13 \Rightarrow x = \frac{4}{5}$ (Not possible)

Case-2: $\frac{1}{5} \leq x < 2$

$-3(x - 2) - (1 - 5x) + 4(3x + 1) = 13 \Rightarrow x = \frac{2}{7}$ (Possible)

Case-3: $-\frac{1}{3} \leq x < \frac{1}{5}$

$-3(x - 2) + (1 - 5x) + 4(3x + 1) = 13 \Rightarrow x = \frac{1}{2}$ (Not possible)

Case-4: $x < -\frac{1}{3}$

$-3(x - 2) + (1 - 5x) - 4(3x + 1) = 13 \Rightarrow x = -\frac{1}{2}$ (Possible)

132. $\log_{\cos x} \sin x \geq 2 \Rightarrow \sin x \leq \cos^2 x$

$\sin^2 x + \sin x - 1 \leq 0$

$$0 < \sin x \leq \frac{\sqrt{5}-1}{2} \qquad (\sin x > 0)$$

133. Minimum value $\frac{-D}{4} = -5 \Rightarrow D = 20$

$$|\alpha - \beta| = \frac{\sqrt{D}}{1} = \sqrt{20}$$

134. $|x-3| + |x+5| = 7x$

$2x + 2 = 7x \qquad x \geq 3$

$-(x-3) + (x+5) = 7x \qquad -5 < x < 3$

$-(x-3) - (x+5) = 7x \qquad x \leq -5$

136. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ac) \Rightarrow ab + bc + ac = -4$

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) \Rightarrow abc = -4$

140. $x^2 - 3x + 4 < x^2 + 3x + 4$

$\Rightarrow \qquad x > 0$

142. $x^2 + 4x + 3 = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$\alpha = -3, \beta = -1$

143. $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \quad a + b + c = 0$

$\Rightarrow \quad ax^2 + bx + c = 0$ has one root $x = 1$

145. $x_1 + x_2 + x_1 x_2 = a$

$x_1 x_2 + x_1 x_2 (x_1 + x_2) = b$

$x_1^2 x_2^2 = c \quad \Rightarrow b + c = x_1 x_2 (a+1)$

147. $(|x|-2)(|x|-1) = 0 \quad \Rightarrow x = \pm 1, \pm 2$

149. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 4(1 - \sin 2\theta)^2 + 4\cos^2 2\theta$

$= 4(2 - 2\sin 2\theta)$

150. $\sin^2 x + \sin x = -b \; \forall \; x \in [0, \pi]$

$0 \leq -b \leq 2$

$-2 \leq b \leq 0$

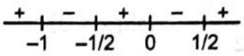
152. $x^2 + px - r = 0 = (x - \gamma)(x - \delta)$

$\alpha^2 + p\alpha - r = (\alpha - \gamma)(\alpha - \delta) = -q - r$

153. $2^{x+2} - 4^x \leq 9 \cap 2^{x+2} - 4^x > 0; \; 2^x(4 - 2^x) > 0$

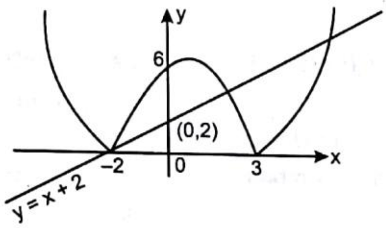
Exercise-2 : One or More than One Answer is/are Correct

1. $\dfrac{2x-1}{x(x+1)(2x+1)}>0$



$\Rightarrow \quad x\in(-\infty,-1)\cup\left(-\dfrac{1}{2},0\right)\cup\left(\dfrac{1}{2},\infty\right)$

3.



4. Apply $D\geq 0\cap f(2)>0\cap f(-2)>0\cap -2<\dfrac{a}{2}<2$

5. $f(x)=x-3 \qquad x>4$

$=5-x \qquad 2<x<4$

$=x+1 \qquad 1<x<2$

$=3-x \qquad x<1$

6. $a-b=\dfrac{1}{b}-\dfrac{1}{a} \qquad \Rightarrow ab=1 \qquad (\because a\neq b)$

$a-b=\dfrac{a}{b} \qquad \Rightarrow a-\dfrac{1}{a}=a^2 \quad \Rightarrow a^3-a^2+1=0$

7. If $f(2+x)=f(2-x)$ and $D>0$

Vertex of parabola is $\left(2,-\dfrac{D}{4a}\right)$ lies in IV$^{\text{th}}$ quadrant.

$f(0)>f(1)>f(2)$

8. If $f(2+x)=f(2-x)$ and $D<0$

$f(-2)=4a-2b+c>0$

If $\log_{f(2)} f(3)$ is not defined then $f(2)=1$

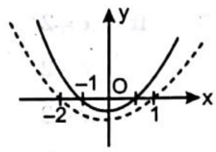
$\Rightarrow \quad f(x)\geq 1$

If $\dfrac{-b}{2a}=2\Rightarrow a$ and b are opposite sign.

9. **Case-I :** $f(-1)\geq 0\cap f(1)<0\cap f(2)\geq 0$

$a\leq 0\cap a<0\cap a\geq -\dfrac{3}{2}$

$\Rightarrow \qquad a\in\left[-\dfrac{3}{2},0\right)$



Case-II: $f(1) \geq 0 \cap f(-1) < 0 \cap f(-2) \geq 0$

$a \geq 0 \cap a > 0 \cap a \leq \frac{3}{2}$

$\Rightarrow \quad a \in \left(0, \frac{3}{2}\right]$

10. As expression taking minimum value

So, $a > 0$

$\frac{-b}{2a} < 0; \quad \frac{-D}{4a} < 0$

$\Rightarrow \quad a > 0, b > 0, D > 0$

11. $ax^2 + bx + c > 0 \; \forall \; x \in R$

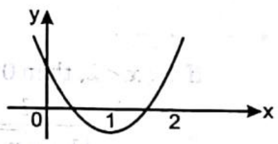
$a > 0, D < 0$

$f(0) = c > 0$

$f(-3) + f(-2) = 13a - 5b + 2c > 0$

$f(-3) + f(2) = 13a - b + 2c > 0$

12. $D > 0 \quad \Rightarrow k > \sqrt{5}$



$f(1) < 0 \quad \Rightarrow k > 3$

$f(2) > 0 \quad \Rightarrow k < \frac{21}{4}$

$\Rightarrow \quad 3 < k < \frac{21}{4}$

13. $x^2 + px + q = 0$

Sum of the roots $= -13$

Product of the roots $= 30$

$\Rightarrow \quad x^2 + 13x + 30 = 0 = (x + 10)(x + 3)$

$\Rightarrow$ Correct roots are $x = -10, -3$

14. $x^2 - 3x + 2 > 0$

$(x - 2)(x - 1) > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

$x^2 - 3x - 4 \leq 0$

$(x - 4)(x + 1) \leq 0 \Rightarrow x \in [-1, 4]$

then $x \in [-1, 1) \cup (2, 4]$

15. $5^x + (2\sqrt{3})^{2x} - 169 \leq 0$

$5^x + 12^x - 169 \leq 0$

if $x=2$ $\quad 5^2+12^2=169$

$x>2$ $\quad 5^x+12^x>169$

$x<2$ $\quad 5^x+12^x<169$

$\Rightarrow \quad x \in (-\infty, 2]$

16. $f(x)=x^2+ax+b$

$D_1 : a^2-4b$

$g(x)=x^2+cx+d$

$D_2 : c^2-4d$

$D_1+D_2=a^2+c^2-4(b+d)=(a-c)^2>0 \Rightarrow$ atleast one of them is positive.

17. Let $x-1=t^2$

$$\frac{1}{\sqrt{x+2\sqrt{x-1}}}+\frac{1}{\sqrt{x-2\sqrt{x-1}}}=\frac{1}{\sqrt{t^2+2t+1}}+\frac{1}{\sqrt{t^2-2t+1}}$$

$$=\frac{1}{|t+1|}+\frac{1}{|t-1|}=\frac{1}{|1+\sqrt{x-1}|}+\frac{1}{|\sqrt{x-1}-1|}$$

If $1<x<2$, then $0<\sqrt{x-1}<1$

$$\frac{1}{|1+\sqrt{x-1}|}+\frac{1}{|\sqrt{x-1}-1|}=\frac{1}{1+\sqrt{x-1}}+\frac{1}{1-\sqrt{x-1}}=\frac{2}{2-x}$$

If $x>2$, then $\sqrt{x-1}>1$

$$\frac{1}{|1+\sqrt{x-1}|}+\frac{1}{|\sqrt{x-1}-1|}=\frac{1}{\sqrt{x-1}+1}+\frac{1}{\sqrt{x-1}-1}=\frac{2\sqrt{x-1}}{x-2}$$

18. $\log_{1/3}(x^2+2px+p^2+1)\geq 0$

$\Rightarrow \quad (x+p)^2+1\leq 1 \;\Rightarrow (x+p)^2\leq 0 \;\Rightarrow x=-p$

$kp^2-kp-k^2\leq 0\;\forall\, k\in R$

$k^2+(p-p^2)\geq 0\;\forall\, k\in R$

$D\leq 0$

19. (a) $\alpha+\beta=\alpha^2+\beta^2$

and $\alpha\beta=\alpha^2\beta^2 \Rightarrow \alpha\beta(\alpha\beta-1)=0 \Rightarrow \alpha=0$ or $\beta=0$ or $\alpha\beta=1$

(b) $\tan 2\theta+\tan 3\theta=\dfrac{\sin 5\theta}{\cos 2\theta\cos 3\theta}=0 \Rightarrow \sin 5\theta=0 \Rightarrow \theta=\dfrac{n\pi}{5}$

(c) $\dfrac{\left(\dfrac{2x_1}{x_2}+\dfrac{128x_3^2}{x_2^2}+\dfrac{x_2^3}{4x_1x_3^2}\right)}{3}\geq 4 \qquad (\because \text{AM} \geq \text{GM})$

(d) Equation of chord with mid-point (h, k) is $T = S_1$

$\Rightarrow \quad (h-1)x + (k-3)y + (h + 3k - h^2 - k^2) = 0$

If it is passes from $(0, 0)$.

Then, $h^2 + k^2 - h - 3k = 0$

20. $-2 < a < 2 \quad \Rightarrow a^2 \in [0, 4)$

$x^2 - 4x - a^2 = 0 \Rightarrow x = 2 \pm \sqrt{4 + a^2}$

21. If $\alpha + 2\beta = 0$

$\Rightarrow \alpha\beta < 0 \Rightarrow -2\beta^2 < 0 \Rightarrow q < 0$

$\alpha + \beta = -\beta = p$

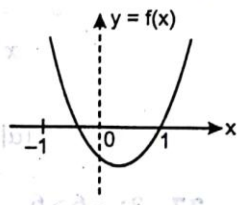
$\alpha\beta = -2\beta^2 = q \Rightarrow 2p^2 + q = 0$

22. $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0 \Rightarrow f(1) = 0$

(a) if $a > b > c > 0$

$\Rightarrow \quad a + b > 2c$

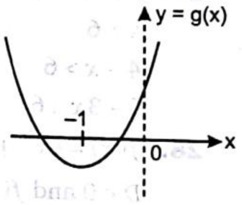
$f(0) = a + c - 2b < 0$



(c) $g(x) = ax^2 + 2bx + c = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$g(0) = c > 0$

$g(-1) = a - 2b + c < 0$



(d) $cx^2 + 2bx + a = 0 \begin{cases} 1/\alpha \\ 1/\beta \end{cases}$

23. $f(x) = 4x^2 - 8ax + a$

$D = 48a^2 \geq 0$

(a) If $f(x)$ is non-negative $\forall\ x \in R$, then $a = 0$

(b) If $a < 0$, then $f(0) < 0$

(c) If $f(x) = 0$ has two distinct solutions in $(0, 1)$, then

$$f(0) > 0 \Rightarrow a > 0$$

$$f(1) > 0 \Rightarrow a < \frac{4}{7}$$

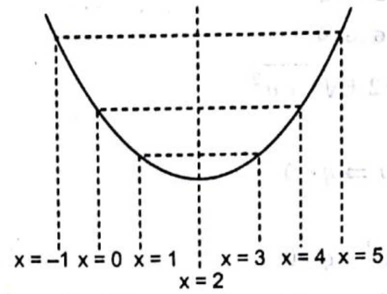
$$0 < \frac{-b}{2a} < 1 \Rightarrow 0 < a < 1$$

24. $ax^2 + bx + c = 0$ has no real roots, then $D < 0$

$$f\left(-\frac{1}{2}\right) = a - 2b + 4c > 0 \Rightarrow a > 0$$

$$\frac{4a+2b+c}{a+3b+9c}=\frac{f(2)}{f\left(\frac{1}{3}\right)}>0$$

25.



26. $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm i\sqrt{4ac-b^2}}{2a}$$

$$|\alpha|=|\beta|=\sqrt{\left(\frac{-b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}=\sqrt{\frac{c}{a}}$$

27. $3x-6>6$ $\qquad x>3$

$x>6$ $\qquad 2\le x\le 3$

$4-x>6$ $\qquad 1\le x<2$

$6-3x>6$ $\qquad x<1$

28. $f(x)=ax^2+x+b-a$

$D<0$ and $f(1)=b+1>0$

$f(0)=b-a>0$

$f(1/2)=4b+2-3a>0$

29.

$$a^2+b^2=(a+b)^2-2ab=7 \qquad \text{...(1)}$$

$$a^3+b^3=(a+b)(7-ab)=10 \qquad \text{...(2)}$$

$\Rightarrow \quad (a+b)[21-(a+b)^2]=20$

Let $\qquad (a+b)=x$

$$x^3-21x+20=0$$

$$(x-1)(x+5)(x-4)=0$$

31. $\alpha+\beta+\gamma+\delta=0$

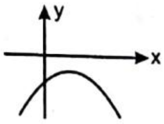
Root $=-\frac{1}{\delta},-\frac{1}{\gamma},-\frac{1}{\beta},-\frac{1}{\alpha}$

Put $x\to-\frac{1}{x}$

32. $D \geq 0 \;\cap\; f(-1) > 0 \;\cap\; f(1) > 0$

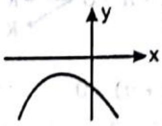
$-2 < K \leq \frac{1}{4}$

33. (a)



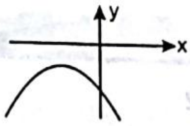
$a < 0$
$c < 0$
$b < 0$

(b)



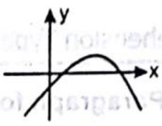
$a < 0$
$c < 0$
$b > 0$

(c)



$a < 0$
$c > 0$
$b > 0$

(d)



$a < 0$
$c < 0$
$b < 0$

34. (a) $f(1)f(-1) > 0$ (b) $f(1)f\left(-\frac{1}{2}\right) > 0$

(c) $f(-1)f(-2) > 0$ (d) $b^2 - 4ac < 0$

but a can be +ve or –ve.

35. $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = \frac{b^2 - 2ac}{a^2}$$

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)}{(\alpha\beta)^3} = \frac{-b(b^2-3ac)}{c^3}$$

36. $\lambda = \sin^2 x + \sin x - 1$

$-1 < \lambda < 1$

37. $x^2 + 5x = x + a + 3 \;\forall\; x \in (-5, 0)$

$x^2 + 4x - 3 = a \;\forall\; x \in (-5, 0)$

$\Rightarrow \quad a \in (-3, 2]$

39. $x^2 - 2ax - a^2 = 0 \qquad x > a$

$\Rightarrow \quad x = a(1 \pm \sqrt{2})$

$x^2 + 2ax - 5a^2 = 0 \qquad x < a$

43. $(\alpha+\beta) + (\gamma+\delta) = 12 \quad \Rightarrow \alpha + \beta = \gamma + \delta = 6$ ...(1)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = 54 \quad \Rightarrow \quad \alpha\beta + \gamma\delta = 9$ ...(2)

$(\alpha\beta)(\gamma\delta) = 14$ ...(3)

$\Rightarrow \quad \alpha\beta = 7,\ \gamma\delta = 2$

44. $l\left(\dfrac{K^3}{K-1}\right) + m\left(\dfrac{K^2-3}{K-1}\right) + n = 0$ → $K = a$, $K = b$, $K = c$

$lK^3 + mK^2 + nK - (3m + n) = 0$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $f(-1-x) = f(-1+x)\ \forall\ x \in R$

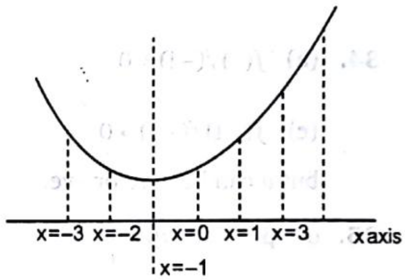
$\Rightarrow$ graph of $f(x)$ is symmetric about $x = -1$.

$$-\frac{b}{2a} = -1 \quad \Rightarrow \quad b = 2a$$

$$\alpha = f(-2) = 4a - 2b + c$$

$$\beta = f(3) = 9a + 3b + c$$

$$\gamma = f(-3) = 9a - 3b + c$$



Using graph $f(3) > f(-3) > f(-2) \Rightarrow \beta > \gamma > \alpha$

2. $p = b - 4a = -2a < 0$

$q = 2a + b = 4a > 0$

$\Rightarrow \quad p \times q < 0$

Paragraph for Question Nos. 3 to 4

Sol. $(k+1)x^2 - (20k+14)x + 91k + 40 = 0$

$\left.\begin{array}{l} f(4) = 27k > 0 \\ f(7) = -9 < 0 \end{array}\right] \rightarrow$ One root is lie (4, 7)

$\left.\begin{array}{l} f(10) = -9k < 0 \\ f(13) = 27 > 0 \end{array}\right] \rightarrow$ Other root is lie (10, 13)

Paragraph for Question Nos. 5 to 7

5. $f(x) = x^2 + bx + c\ \forall\ x \in R$

Least value at $\dfrac{-b}{2} = -1 \Rightarrow b = 2$

Graph of $f(x)$ cuts y-axis, when $x = 0$

$\Rightarrow \quad c = 2$

$\Rightarrow \quad f(x) = x^2 + 2x + 2$

Least value of $f(x) = 1$

6. $f(-2) + f(0) + f(1) = 9$

7. $a \in (1, \infty)$

Paragraph for Question Nos. 8 to 9

Sol. $(\log_2 x)^2 - 4(\log_2 x) - m^2 - 2m - 13 = 0$

8. $D > 0 \Rightarrow m^2 + 2m + 17 > 0 \; \forall \; m \in R$

9. $m^2 + 2m - (\log_2 x)^2 + 4(\log_2 x) + 13 = 0$

$D \geq 0$

$\Rightarrow \quad (\log_2 x - 6)(\log_2 x + 2) \geq 0 \Rightarrow x \in \left(0, \frac{1}{4}\right] \cup [64, \infty)$

Paragraph for Question Nos. 10 to 11

Sol. $x^4 - 2x^3 - 3x^2 + 4x - 1 = 0$ has four roots $a, \frac{1}{a}, b, \frac{-1}{b}$

$$\left(a + \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) = 2 \qquad \ldots(1)$$

$$\left(b - \frac{1}{b}\right) - \left(a + \frac{1}{a}\right) = -4 \qquad \ldots(2)$$

Paragraph for Question Nos. 12 to 14

Sol. $f(x) - (6 - x) = 0 = (x-1)(x-2)(x-3)(x-4)(x-5)$

$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5) + (6 - x)$

Paragraph for Question Nos. 15 to 16

Sol. $x^3 - x^2(1 + \sin\theta + \cos\theta) + x(\sin\theta + \cos\theta + \sin\theta\cos\theta) - \sin\theta\cos\theta = 0$

$\Rightarrow$ Roots are $1, \sin\theta, \cos\theta$.

Paragraph for Question Nos. 17 to 18

Sol. $2[1 + P(x)] = P(x-1) + P(x+1)$

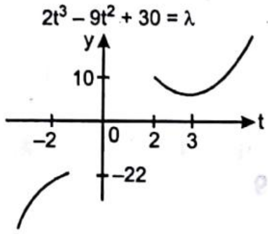
$2 + 2[ax^2 + bx + c] = a(x-1)^2 + b(x-1) + c + a(x+1)^2 + b(x+1) + c$

$\Rightarrow \quad a = 1$

$P(0) = c = 8$

$P(2) = 4a + 2b + c = 32 \Rightarrow b = 10$

Paragraph for Question Nos. 19 to 21

Sol. 

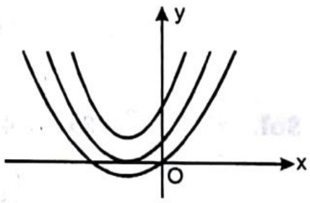
Paragraph for Question Nos. 22 to 23

22. $D > 0$

$(2t-1)^2 - 4t(5t-1) > 0$

$16t^2 - 1 < 0 \Rightarrow \dfrac{-1}{4} < t < \dfrac{1}{4} \qquad (t \neq 0)$

23. $t > 0$



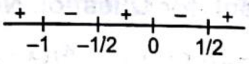
Case-I: $D < 0$

$(2t-1)^2 - 4t(5t-1) < 0 \Rightarrow t > \dfrac{1}{4} \quad (\because t > 0)$

Case-II: $D \geq 0 \cap f(0) \geq 0 \cap \dfrac{-b}{2a} \leq 0 \;\; \Rightarrow \dfrac{1}{5} \leq t \leq \dfrac{1}{4}$

Exercise-4 : Matching Type Problems

1. (A) $\dfrac{(2x-1)}{x(2x+1)(x+1)} > 0$



(B) $3x^2 + 2(a^2+1)x + (a^2 - 3a + 2) = 0$

$\alpha\beta < 0$

$a^2 - 3a + 2 < 0 \Rightarrow (a-1)(a-2) < 0$

(C) $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$

Let $x - 1 = t^2$; $|t-2| + |t-3| = 1$

(D) A.M. $= \dfrac{\alpha+\beta+\gamma+\delta}{4} = 2$

G.M. $= (\alpha\beta\gamma\delta)^{1/4} = 2$

A.M. = G.M. $\Rightarrow \alpha = \beta = \gamma = \delta = 2$

3. (A) $x^4 - 8x^2 - 9 = 0$

$(x^2 - 9)(x^2 + 1) = 0 \quad \Rightarrow x = 3, -3$

(B) $x^{2/3} + x^{1/3} - 2 = 0$

$(x^{1/3} + 2)(x^{1/3} - 1) = 0 \quad \Rightarrow x = -8, 1$

(C) $(\sqrt{3x+1})^2 = (\sqrt{x} - 1)^2$

$\Rightarrow \quad 3x + 1 = x + 1 - 2\sqrt{x} \quad \Rightarrow 2x = -2\sqrt{x}$ (Not possible)

(D) $(3^x - 9)(3^x - 1) = 0 \quad \Rightarrow x = 0, 2$

4. (A) $\therefore \quad (a+b) = -a \;\&\; ab = b \Rightarrow (1, -2)$ and $(0, 0)$

(B) $P = \overline{O}, \quad Q = 8\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = 1$

(C) $ar^6 = \sqrt{2}$

Product $= (\sqrt{2})^{11} = 2^{11/2}$

$\therefore \quad m = 11$

$n = 4$

(D) $x = y = 3$

$\because \quad (x-y)^2 + (y-3)^2 = 0$

$\therefore \quad 5x - 4y = 3$

Exercise-5 : Subjective Type Problems

1. $f\left(\cos\frac{\pi}{7}\right) = \sin\frac{\pi}{7}\sin\frac{3\pi}{7} + \sin\frac{3\pi}{7}\sin\frac{5\pi}{7} + \sin\frac{\pi}{7}\sin\frac{5\pi}{7}$

$= 2\cos^2\frac{\pi}{7} + \cos\frac{\pi}{7} - 1$

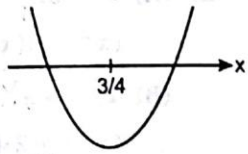
$\Rightarrow \quad f(x) = 2x^2 + x - 1$

2. $(r-a)(r-b)(r-c)(r-d) = (-1)\times(-3)\times(1)\times(3)$

$\Rightarrow \quad (r-a) + (r-b) + (r-c) + (r-d) = 0$

3. Let $x^2 + x + 1 = t \quad \forall\, t \in \left[\frac{3}{4}, \infty\right)$

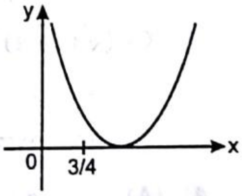
$t^2 - (m-3)t + m = 0$



Case-I : $f\left(\frac{3}{4}\right)<0$

$\frac{9}{16}-\frac{3}{4}(m-3)+m<0$

$\Rightarrow \quad m<\frac{-45}{4}$



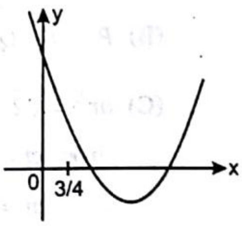
Case-II : $D=0 \Rightarrow m=1,9$

$\frac{-b}{2a}>\frac{3}{4} \Rightarrow m>\frac{9}{2}$

There is one positive integral value of $m=9$.

4. $t^2-(m-3)t+m=0$

$t\in[3/4,\infty)$ has four distinct real roots, then



$D>0$

$\Rightarrow \quad m^2-10m+9>0$

$\Rightarrow \quad m\in(-\infty,1)\cup(9,\infty)$

$\frac{-b}{2a}>\frac{3}{4} \quad \Rightarrow m>\frac{9}{2}$

$f\left(\frac{3}{4}\right)>0 \Rightarrow m>\frac{-45}{4} \quad \Rightarrow m\in(9,\infty)$

5. $f(t)=(m^2-12)t^2-8t-4=0 \qquad (t\geq 0)$

$f(0)=-4<0$

$m^2-12\leq 0 \quad \Rightarrow m\in[-2\sqrt{3},2\sqrt{3}]$

Case-I : $D<0$

$\Rightarrow \quad m^2-8<0 \quad \Rightarrow m\in(-2\sqrt{2},2\sqrt{2})$

Case-II : $D\geq 0 \quad \Rightarrow m\in(-\infty,-2\sqrt{2}]\cup[2\sqrt{2},\infty)$

$\frac{-b}{2a}=\frac{4}{m^2-12}<0 \quad \Rightarrow m\in(-2\sqrt{3},2\sqrt{3})$

$\Rightarrow \quad m\in[-2\sqrt{3},2\sqrt{3}]$

6. $(e^x-2)\left[\sin\left(x+\frac{\pi}{4}\right)\right](x-\ln 2)(\sin x-\cos x)<0$

$\frac{1}{\sqrt{2}}(x-\ln 2)\cdot(\sin x+\cos x)(x-\ln 2)(\sin x-\cos x)<0=\frac{1}{\sqrt{2}}(x-\ln 2)^2(\sin^2 x-\cos^2 x)<0$

$\Rightarrow \quad \cos 2x>0, \quad x\neq \ln 2$

$x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right] - \{\ln 2\}$

Least positive integral value is 3.

7. $x^2 + 17x + 71 = \lambda^2 \Rightarrow \lambda \in Z$

$x^2 + 17x + (71 - \lambda^2) = 0$

$D = \text{perfect square} = m^2 \text{ (say)}$

$m^2 = 289 - 4(71 - \lambda^2)$

$(m - 2\lambda)(m + 2\lambda) = 1 \times 5$

$\Rightarrow \quad m - 2\lambda = 1$

$m + 2\lambda = 5$

8. $P(x) = (x^4 - x^3 - x^2 - 1)(x^2 + 1) + (x^2 - x + 1)$

$P(\alpha) + P(\beta) + P(\gamma) + P(\delta) = (\alpha^2 - \alpha + 1) + (\beta^2 - \beta + 1) + (\gamma^2 - \gamma + 1) + (\delta^2 - \delta + 1) = 6$

9. If $-\frac{a}{2} \le 1$

$f(x)_{\max} = f(4) \Rightarrow 4a + 18 = 6 \Rightarrow a = -3$ (Not possible)

if $-\frac{a}{2} \ge 1$

$f(x)_{\max} = f(-2) \Rightarrow a = 0$ (Not possible)

There is no real value of 'a'.

10. $x^2 - 8x - (n^2 - 10n) = 0$

$D < 0 \quad \Rightarrow n^2 - 10n + 16 < 0$

$(n - 8)(n - 2) < 0$

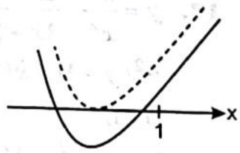
$\Rightarrow 2 < n < 8$ and $n \ne 10$

11. $x^2 + 2(m - 1)x + (m + 5) > 0 \ \forall \ (x > 1)$

Case-I : $D < 0$

$m^2 - 3m - 4 < 0 \Rightarrow -1 < m < 4$

Case-II : $D \ge 0$



$\Rightarrow \quad m \in (-\infty, -1] \cup [4, \infty)$

$f(1) \ge 0 \Rightarrow m \ge -\frac{4}{3}$

$\frac{-b}{2a} < 1 \Rightarrow m > 0$

$\Rightarrow \quad m \in (-1, \infty)$

12. $ax^4 + bx^3 - x^2 + 2x + 3 = (x+2)(x-1)\,Q(x) + (4x+3)$

Put $x = 1$ $\quad a + b = 3$

$x = -2$ $\quad b = 2a$

13. $D > 0 \quad \cap \quad \dfrac{-b}{2a} > 4 \quad \cap \quad f(4) \ge 0$

$k - 1 > 0 \quad \cap \quad 4k > 4 \quad \cap \quad k^2 - 3k + 2 \ge 0$

$k > 0 \quad \cap \quad k > 1 \quad \cap \quad (k-2)(k-1) \ge 0$

$\Rightarrow \quad k \ge 2$

14. $x^2 - 3x + 2 = (x-1)(x-2)$

If $(x-1)$ is a factor of $x^4 - px^2 + q = 0$. Then

$$p - q = 1 \qquad \text{...(1)}$$

If $(x-2)$ is a factor of $x^4 - px^2 + q = 0$. Then

$$4p - q = 16 \qquad \text{...(2)}$$

$\Rightarrow \quad p = 5, q = 4$

$\Rightarrow \quad p + q = 9$

15. $x^2 + 2xy + ky^2 + 2x + k = 0$

if it can be resolved into two linear factors, then

$$abc + 2fgh - bg^2 - af^2 - ch^2 = 0$$

$$k^2 - k - k = 0$$

$$k = 0, 2$$

16. $(a+1)x^2 + 2 = ax + 3$ has exactly one solution.

$\Rightarrow \quad D = 0$

$a^2 + 4(a+1) = 0$

$(a+2)^2 = 0 \quad \Rightarrow a = -2 \quad \Rightarrow a^2 = 4$

17. $y = \dfrac{x^2 - ax + 1}{x^2 - 3x + 2}$

$x^2(y-1) - x(3y-a) + 2y - 1 = 0 \ \forall \ x \in R$

$D \ge 0$

$(3y-a)^2 - 4(y-1)(2y-1) \ge 0 \ \forall \ y \in R$

$y^2 - 6y(a-2) + a^2 - 4 \ge 0 \ \forall \ y \in R$

$D \le 0$

$36(a-2)^2 - 4(a^2-4) \le 0$

$(a-2)(2a-5) \le 0$

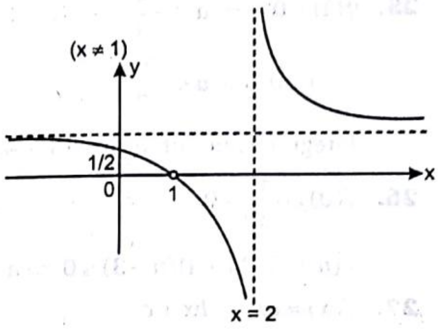
$$2 \le a \le \frac{5}{2}$$

$\Rightarrow$ Integral value of $a = 2$

At $a = 2$

$$f(x) = \frac{x^2 - 2x + 1}{x^2 - 3x + 2} = \frac{(x-1)^2}{(x-2)(x-1)}$$

$$f(x) = \frac{x-1}{x-2}(x \ne 1)$$



Range $R - \{0, 1\}$

$\Rightarrow$ No integral values of 'a' for which range is R.

18. $x^{100} = (x^2 - 3x + 2) \cdot Q(x) + (ax + b)$

at $x = 1 \quad \Rightarrow \quad a + b = 1$ ...(1)

at $x = 2 \quad \Rightarrow \quad 2a + b = 2^{100}$ ...(2)

$\Rightarrow \quad a = 2^{100} - 1, \; b = 2 - 2^{100}$

Remainder $= (2^{100} - 1)\, x + 2(1 - 2^{99})$

$\Rightarrow \quad k = 99$

19. $x = 7^{1/3} + 7^{2/3}$

$x^3 = 7 + 49 + 3 \times 7(x) \;\Rightarrow\; x^3 - 21x - 56 = 0$

Product of all roots $= 56$

21. Clearly $P(x)$ is a second degree polynomial.

$\therefore \quad P(x) = ax^2 + bx + c$

$P'(x) = 2ax + b$

$P(x) - P'(x) = ax^2 + (b - 2a)\,x + c - b = x^2 + 2x + 1$

$a = 1, b - 2a = 2, c - b = 1$

$a = 1, b = 4, c = 5$

$P(x) = x^2 + 4x + 5$

$P(-1) = 1 - 4 + 5 = 6 - 4 = 2$

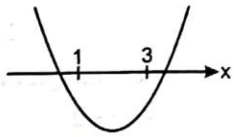
23. Let $x^2 = t$

$t^2 + kt + k = 0$

$D > 0 \quad \Rightarrow k \in (-\infty, 0) \cup (4, \infty)$

$f(0) < 0 \;\Rightarrow k < 0$

25. $f(1) < 0 \Rightarrow a < -\frac{4}{3}$



$f(3) < 0 \Rightarrow a < -\frac{8}{7}$

Integral values of 'a' are $-5, -4, -3, -2$.

26. $f(0) f\left(\frac{\pi}{2}\right) \le 0$

$-(n+1)(2n+1)(n-3) \le 0 \Rightarrow n \in [3, \infty)$

27. $f(x) = ax^2 + bx + c$ $\qquad a, b, c \in I$

$ax^2 + bx + c = a(x-\alpha)(x-\beta) + p$ $\qquad \alpha, \beta \in I$

$ax^2 + bx + c - 2p = a(x-\alpha)(x-\beta) - p = 0$

Not possible for integral values of x.

28. $9x^2 + 2x(y-46) + y^2 - 20y + 244 = 0$

$D \ge 0 \quad \Rightarrow y^2 - 11y + 10 \le 0$

$(y-1)(y-10) \le 0 \Rightarrow 1 \le y \le 10$

$y^2 + 2y(x-10) + 9x^2 - 92x + 244 = 0$

$D \ge 0 \quad \Rightarrow x^2 - 9x + 18 \le 0$

$(x-3)(x-6) \le 0 \Rightarrow 3 \le x \le 6$

29. $a + b = 3$ and $a^3 + b^3 = 7 \Rightarrow a^3 + (3-a)^3 = 7 \Rightarrow 9a^2 - 27a + 20 = 0$

Sum of distinct values of 'a' is 3.

30. $(y^2 - 3)^2 + (x-4)^2 = 1$

$\Rightarrow \quad x = 4 + \cos\theta, \quad y^2 = 3 + \sin\theta$

$M = 36, m = 1$

31. $x_1 + x_2 + x_1x_2 = a$

$x_1x_2 + x_1x_2(x_1 + x_2) = b$

$x_1^2 x_2^2 = c$

If $\quad b + c = 2(a+1) \Rightarrow x_1x_2 = 2$

32. $x^3 + 3x^2 + 4x + 5 = 0 \quad \Rightarrow x = \alpha$ is root

$x^3 - 3x^2 + 4x - 5 = 0 \quad \Rightarrow x = \beta$ is root

$\Rightarrow \qquad \alpha + \beta = 0$

33. 5

$(1) - (2) \qquad\qquad (2) - (3)$

$(x-z)(x+y+z) = 1$ and $(y-x)(x+y+z) = -2$

Divide $z = \frac{x+y}{2}$

$$y^2 + y\left(\frac{x+y}{2}\right) + \left(\frac{x+y}{2}\right)^2 = 1 \text{ and } x^2 - 2xy - 5y^2 = 0$$

$$x = (1+\sqrt{6})y$$

$y^2 = \frac{2}{9+3\sqrt{6}}$ put values

34. $\frac{4(1-a-b)-(a-b)^2}{4} > \frac{4(1+a+b)-(a+b)^2}{-4}$

$\Rightarrow \quad 8 > (a+b)^2 + (a-b)^2 \Rightarrow a^2 + b^2 < 4$

35. $\sqrt[3]{20x + \sqrt[3]{20x+13}} = 13$

$$\sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + \ldots\infty}}}}} = 13$$

$\Rightarrow \quad \sqrt[3]{20x+13} = 13 \quad \Rightarrow 20x = 2197 - 13$

$\Rightarrow \quad x = \frac{2184}{20} = \frac{546}{5}$

36. Let $f(x) = x^2 - 2(a+1)x + a(a-1)$

$f(1-a) < 0 \quad \cap \quad f(1+a) < 0$

$4a^2 - 3a - 1 < 0 \quad \cap \quad 3a + 1 > 0$

$-\frac{1}{4} < a < 1 \quad \cap \quad a > -\frac{1}{3}$

$\Rightarrow \quad a \in \left(-\frac{1}{4}, 1\right)$

37. $(x-8)(x-2) < 0$

$\Rightarrow \quad 2 < x < 8$

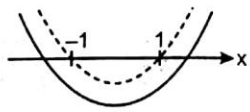
38. $\sin\theta + \cos\theta = \frac{-b}{a}$

$\sin\theta \cdot \cos\theta = \frac{c}{a}$

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta = \frac{b^2}{a^2} = 1 + \frac{2c}{a}$

$\Rightarrow \quad \frac{b^2 - a^2}{a^2} = \frac{2c}{a}$

39. $\cos^2 x + (1-a)\cos x - a^2 \le 0 \ \forall\ x \in R$



Let $\cos = t \ \forall\ t \in [-1,1]$

$$t^2 + (1-a)t - a^2 \le 0 \ \forall\ t \in [-1,1]$$

$$f(-1) \le 0$$

$\Rightarrow \quad a^2 - a \ge 0$

$\Rightarrow \quad a \in (-\infty, 0] \cup [1, \infty)$

$$f(1) \le 0$$

$\Rightarrow \quad a^2 + a - 2 \ge 0$

$(a+2)(a-1) \ge 0 \Rightarrow a \in (-\infty, -2] \cup [1, \infty)$

40. $2x^2 - 35x + 2 = 0$ (roots α, β)

$2\alpha - 35 = -\dfrac{2}{\alpha}$ and $2\beta - 35 = -\dfrac{2}{\beta}$

42. $xF(x) - 1 = k(x-1)(x-2)(x-3)\ldots(x-9)$

$\Rightarrow \quad F(x) = \dfrac{k(x-1)(x-2)(x-3)\ldots(x-9)+1}{x}$

Constant term $= k(-9!) + 1 = 0$

$\Rightarrow \quad k = \dfrac{1}{9!}$

44. $\cos A + \cos B + \cos C = -a$

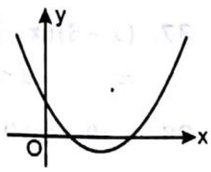
$\cos A \cos B + \cos B \cos C + \cos A \cos C = b$

$$\cos A \cos B \cos C = -c$$

$$a^2 - 2b - 2c = \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C$$

$$= 1$$

45. $k > 0$



$D > 0 \cap \dfrac{-b}{2a} > 0$

$k^2 - 10k + 9 > 0 \cap \dfrac{k-3}{k} < 0$

$k \in (-\infty, 1) \cup (9, \infty) \cap k \in (0, 3) \Rightarrow k \in (0, 1)$

❑❑❑

Solution to Chapter 9 till end (Chapter 26) is in part 2

Balaji

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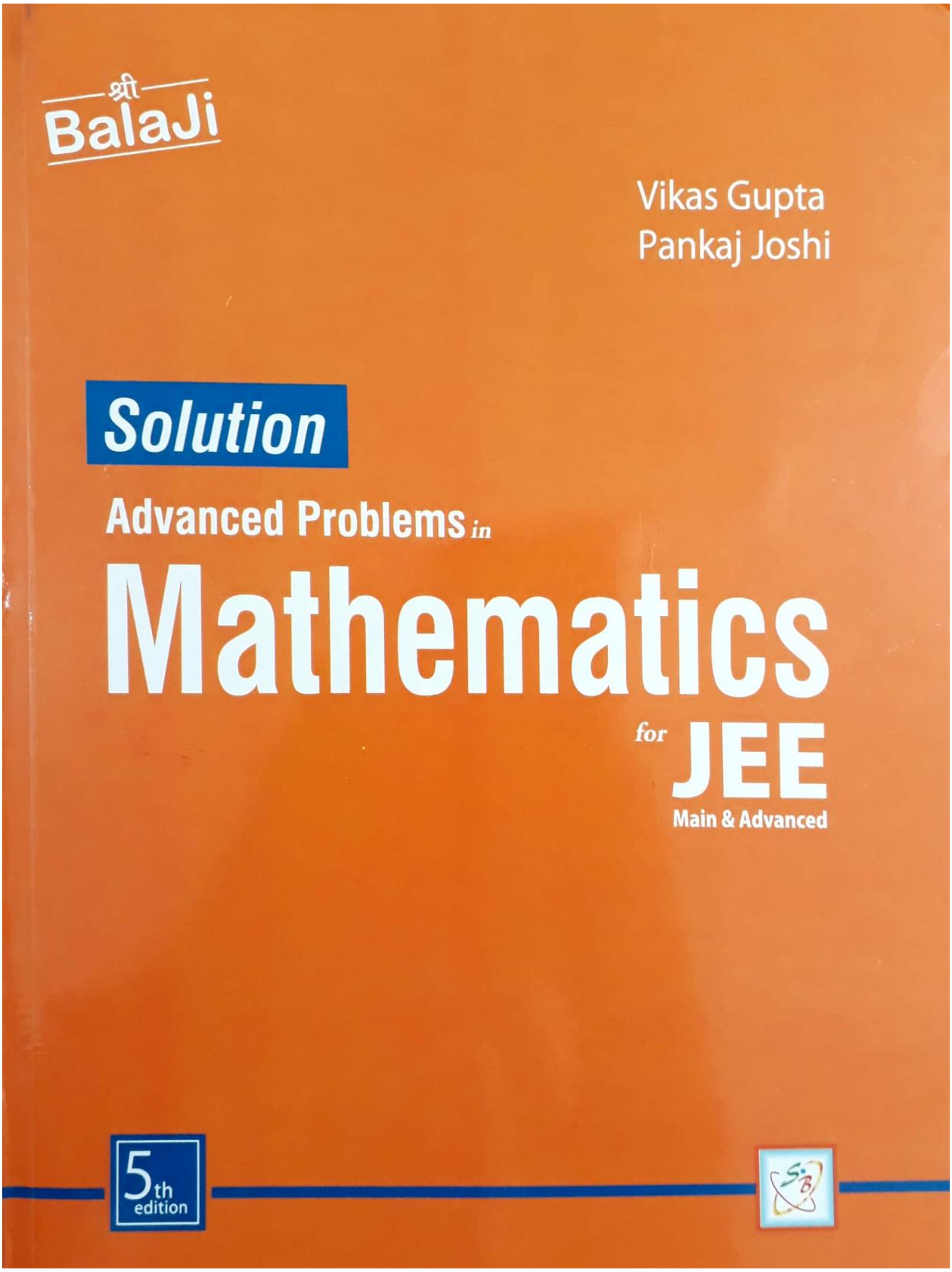
for IIT JEE Main and Advanced

by

Vikas Gupta and Pankaj Joshi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Chapter 9 – Sequence and Series

9 SEQUENCE AND SERIES

Exercise-1 : Single Choice Problems

1. AM $\geq$ GM

3. $2\sec\alpha = \sec(\alpha-2\beta)+\sec(\alpha+2\beta)$

$$\frac{2}{\cos\alpha}=\frac{\cos(\alpha+2\beta)+\cos(\alpha-2\beta)}{\cos(\alpha-2\beta)\cos(\alpha+2\beta)}$$

$$\cos 2\alpha+\cos 4\beta=\cos\alpha\,(2\cos\alpha\cos 2\beta)$$

$$2\cos^2\alpha-1+2\cos^2 2\beta-1=2\cos^2\alpha\cos 2\beta$$

$\Rightarrow$ $\cos^2\alpha(1-\cos 2\beta)+(\cos 2\beta+1)(\cos 2\beta-1)=0$

$\Rightarrow$ $\cos^2\alpha=1+\cos 2\beta$

4. If a, b, c A.P. $\Rightarrow$ $b=\dfrac{a+c}{2}$

if c, d, e H.P. $\Rightarrow$ $d=\dfrac{2ec}{e+c}$

if b, c, d G.P. $\Rightarrow$ $c^2=bd$

$$c^2=\left(\frac{a+c}{2}\right)\left(\frac{2ec}{e+c}\right)$$

$\Rightarrow$ $c^2=ae$

5. $(a+nd)^2=(a+md)(a+rd)$

$\Rightarrow$ $\dfrac{a}{d}=\dfrac{mr-n^2}{2n-m-r}$

if m, n, r in H.P., then $n=\dfrac{2mr}{m+r}$ $\Rightarrow$ $\dfrac{a}{d}=\dfrac{-n}{2}$

6. A.M. $(\alpha,\beta,\gamma,\delta)=\dfrac{4}{4}=1$

G.M. $(\alpha,\beta,\gamma,\delta)=1$ $\Rightarrow$ $\alpha=\beta=\gamma=\delta=1$

So, equation is $(x-1)^4=0$

7. $S_3 = S_1^2 \Rightarrow \dfrac{S_1^4 S_2^2 - S_2^2 S_3^2}{S_1^2 + S_3^2} = \dfrac{S_2^2(S_1^4 - S_3^2)}{S_1^2 + S_3^2} = 0$

8. $T_r = \dfrac{r \cdot 2^r}{(r+2)!}$

$$T_r = \frac{(r+2-2)2^r}{(r+2)!} = \frac{1}{(r+1)!}2^r - \frac{1}{(r+2)!}2^{r+1}$$

$$S_n = \frac{2!}{2!} - \frac{2^{n+1}}{(n+2)!}$$

$$\lim_{n\to\infty} S_n = S_\infty = 1 \quad \left[\text{as } \lim_{n\to\infty} \frac{2^{n+1}}{(n+2)!} = 0\right]$$

9. $\tan^2 \dfrac{\pi}{12} = \tan\left(\dfrac{\pi}{12} - x\right)\tan\left(\dfrac{\pi}{12} + x\right)$

$$\tan^2 \frac{\pi}{12} = \frac{\tan^2 \frac{\pi}{12} - \tan^2 x}{1 - \tan^2 \frac{\pi}{12} \tan^2 x} \Rightarrow \tan^2 x\left(\tan^4 \frac{\pi}{12} - 1\right) = 0 \Rightarrow \tan x = 0$$

$x = 0, \pi, 2\pi, 3\pi, \ldots\ldots 99\pi$

10. $\dfrac{S_n}{S_n - 1} = \dfrac{n}{n-1}\dfrac{n+1}{n+2}$

$$Q_n = \left(\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \ldots\ldots \frac{n}{n-1}\right) \times \left(\frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \ldots\ldots \times \frac{n}{n+1} \times \frac{n+1}{n+2}\right)$$

$$Q_n = \left(\frac{n}{1}\right) \cdot \left(\frac{3}{n+2}\right) = \frac{3n}{n+2}$$

$$\lim_{n\to\infty} Q_n = 3$$

11. $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$

$$\begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} = 0$$

12. Numbers divisible by $6 \to 49$

Numbers divisible by $18 \to 16$

13. $\dfrac{y+z}{2} = \sqrt{yz} \quad \Rightarrow \quad 1 - x \geq 2\sqrt{yz}$

Thus, $(1-x)(1-y)(1-z) \geq 2\sqrt{yz}\cdot 2\sqrt{zx}\cdot 2\sqrt{xy} = 8xyz$

$\Rightarrow \quad \dfrac{xyz}{(1-x)(1-y)(1-z)} \leq \dfrac{1}{8}$

17. Clearly, both roots are lies in between -1 and 1.

$\therefore \quad \lim_{n\to\infty}\sum_{r=1}^{n}(\alpha^r+\beta^r) = \left(\lim_{n\to\infty}\sum_{r=1}^{n}\alpha^r\right) + \left(\lim_{n\to\infty}\sum_{r=1}^{n}\beta^r\right)$

$= \dfrac{\alpha}{1-\alpha} + \dfrac{\beta}{1-\beta} = \dfrac{1}{12}$

18. $\sum \dfrac{a_i}{a_j} = \dfrac{a_1}{a_2} + \dfrac{a_1}{a_3} + \dfrac{a_1}{a_4} + \dfrac{a_2}{a_1} + \dfrac{a_2}{a_3} + \dfrac{a_2}{a_4} + \dfrac{a_3}{a_1} + \dfrac{a_3}{a_2} + \dfrac{a_3}{a_4} + \dfrac{a_4}{a_1} + \dfrac{a_4}{a_2} + \dfrac{a_4}{a_3}$

$\geq 12 \qquad \left(\because x + \dfrac{1}{x} \geq 2\right)$

19. $\dfrac{x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2}{6} \geq \sqrt[6]{2^2 \cdot 4 \cdot x^4 \cdot y^4 \cdot z^4}$

20. Let first term be 'a' and difference be d.

$\Rightarrow \quad 5(a+4d) = 8(a+7d)$

$\Rightarrow \quad a + 12d = 0$

$S_{25} = \dfrac{25}{2}[2a + 24d]$

$S_{25} = 25(a+12d) = 0$

21. $10\sin x = \sqrt{5}(4\sin^2 x + 1) \qquad \sin x \neq 0$

$\Rightarrow \quad \sin x = \dfrac{\sqrt{5}\pm 1}{4}$

22. Let first term of G.P. be a and ratio be r.

$\Rightarrow \quad a + ar + ar^2 = 70$ and $10ar = 4a + 4ar^2$

$\Rightarrow \quad a = 40 \qquad r = \dfrac{1}{2}$

$S = \dfrac{a}{1-r} = \dfrac{40}{1-\dfrac{1}{2}} = 80$

23. $\sum_{n=1}^{\infty}\dfrac{k}{2^{n+k}} = \dfrac{k}{2^k}\sum_{n=1}^{\infty}\dfrac{1}{2^n} = \dfrac{k}{2^k}$

$\sum_{k=1}^{\infty}\dfrac{k}{2^k} = \dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \dfrac{4}{2^4} + \ldots\ldots = 2$

24. $(pqr)^{1/3} \geq \dfrac{p+q+r}{3} \quad \Rightarrow p = q = r$

if $3p + 4q + 5r = 12 \quad \Rightarrow p = q = r = 1$

25. $\frac{1}{3}\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\ldots\right)=\frac{1}{3}\left[\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{5}\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{7}\left(\frac{1}{3}+\frac{1}{4}\right)+\ldots\right]$

$$=\frac{1}{3}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\right]=\frac{1}{3}$$

26. $\dfrac{\frac{a}{2}[2A+(a-1)D]}{a^2}=\dfrac{\frac{b}{2}[2A+(b-1)D]}{b^2}=c \Rightarrow D=2c, A=c$

27. $\dfrac{x/r}{1-r}=4 \Rightarrow \dfrac{x}{4}=r-r^2$

if $-1<r<1$ then $-2<r-r^2<\frac{1}{4}$

$$-2<\frac{x}{4}<\frac{1}{4} \Rightarrow -8<x<1$$

28. $t_1+t_3+t_5+\ldots+t_{2n+1}=\frac{n+1}{2}[2a+n(2d)]=248$

$t_2+t_4+t_6+\ldots+t_{2n}=\frac{n}{2}[2(a+d)+(n-1)2d]=217$

$t_{2n+1}-t_1=2n\cdot d=56$

$\Rightarrow \quad \frac{n+1}{2}[2a+56]=248$ and $\frac{n}{2}[2a+56]=217$

$\Rightarrow \quad n=7,\ a=3$

29. length of side $A_1=20$

length of side $A_2=\frac{20}{\sqrt{2}}$

length of side $A_3=\frac{20}{(\sqrt{2})^2}$

length of side $A_n=\frac{20}{(\sqrt{2})^{n-1}}$

Area of $A_n=\frac{400}{2^{n-1}}<1$

30. $S_k=\sum_{i=0}^{\infty}\frac{1}{(k+1)^i}=1+\frac{1}{k+1}+\frac{1}{(k+1)^2}+\ldots=\frac{k+1}{k}$

$$\sum_{k=1}^{n}kS_k=\sum_{k=1}^{n}(k+1)=\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=\frac{n(n+1)}{2}+n=\frac{n(n+3)}{2}$$

31. $T_r=\frac{(r^2+1)}{r(r+1)}\cdot 2^{r-1}=\left(1+\frac{1}{r}-\frac{2}{r+1}\right)2^{r-1}$

$$S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} 2^{r-1} + \sum_{r=1}^{n}\left(\frac{2^{r-1}}{r} - \frac{2^r}{r+1}\right) = (2^{n-1}) + \left[1 - \frac{2^n}{n+1}\right] = \left(\frac{n}{n+1}\right)2^n$$

32. $\sum_{n=2}^{29}(1.5)^n = (1.5)^2 + (1.5)^3 + \ldots\ldots + (1.5)^{29}$

$$= (1.5)^2\left[\frac{(1.5)^{28} - 1}{0.5}\right] = 2k - 2(1.5)^2$$

33. $7, A_1, A_2, A_3, \ldots\ldots, A_n, 49$ are in A.P.

$$A_1 + A_2 + A_3 + \ldots\ldots + A_n = \left(\frac{n+2}{2}\right)(7+49) - (7+49)$$

$\Rightarrow \quad \frac{n}{2} \times 56 = 364 \ \Rightarrow n = 13$

34. $\frac{2}{r^2}, \frac{2}{r}, 2, 2r, 2r^2$

35. $S_n = 5n^2 + 4n$

$t_n = S_n - S_{n-1} = 10n - 1$

36. $x^3 + y^3 = (x+y)(x^2 + y^2 - xy) = a\left[b - \left(\frac{a^2 - b}{2}\right)\right] \qquad \left(\because xy = \frac{(x+y)^2 - (x^2+y^2)}{2}\right)$

$$= \frac{3ab}{2} - \frac{a^3}{2}$$

37. $S_1 = \dfrac{1}{1 - \frac{2}{3}} = 3$

$S_2 = \dfrac{3}{1 - \frac{2}{5}} = 5$

$\vdots$

$S_n = \dfrac{2n-1}{1 - \frac{2}{2n+1}} = 2n + 1$

$$\frac{1}{S_1S_2S_3} + \frac{1}{S_2S_3S_4} + \frac{1}{S_3S_4S_5} + \ldots\ldots = \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \frac{1}{7\cdot9\cdot11} + \ldots\ldots$$

$$S_\infty = \sum_{r=1}^{\infty} t_r = \sum_{r=1}^{\infty}\frac{1}{(2r+1)(2r+3)(2r+5)} = \sum_{r=1}^{\infty}\left[\frac{1}{(2r+1)(2r+3)} - \frac{1}{(2r+3)(2r+5)}\right]$$

38. $ar^5, 2, 5, ar^{13}$ are in G.P.

$\Rightarrow \quad (ar^9)^2 = 10$

$t_1 t_2 t_3 \ldots\ldots t_{19} = a^{19} r^{9\times19} = (ar^9)^{19} = 10^{19/2}$

39. A.M. ≥ G.M.

$A+\frac{1}{A}+1\geq 3;\quad B+\frac{1}{B}+1\geq 3;\quad C+\frac{1}{C}+1\geq 3;\quad D+\frac{1}{D}+1\geq 3$

$\left(A+\frac{1}{A}+1\right)\left(B+\frac{1}{B}+1\right)\left(C+\frac{1}{C}+1\right)\left(D+\frac{1}{D}+1\right)\geq 3^4$

40. $(\Sigma r)^2=\Sigma r^2+2\Sigma r_1r_2$

$\Sigma r_1r_2=\frac{a-b}{2}$

41. $\frac{2n}{2}[2a+(2n-1)d]=x$ and $\frac{n}{2}[2(a+2nd)+(n-1)d]=y$

$\Rightarrow \quad \frac{2y}{n}-\frac{x}{n}=3nd \Rightarrow d=\frac{2y-x}{3n^2}$

44. $2, 6, 2(k-1)$ are in G.P.

$\Rightarrow \quad 6^2=2\times 2(k-1)$

$\Rightarrow \quad k=10$

$\Rightarrow \quad x^2-x-6>0$ and $|x|<100$

$\Rightarrow \quad x\in(-100,-2)\cup(3,100)$

Number of integers = 193

45. $\sum_{r=1}^{n}\sqrt{1+T_rT_{r+1}T_{r+2}T_{r+3}}=\sum_{r=1}^{n}\sqrt{1+\left(r-\frac{3}{2}\right)\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)}$

$=\sum_{r=1}^{n}\sqrt{\left(r^2-\frac{5}{4}\right)^2}=\sum_{r=1}^{n}\left|r^2-\frac{5}{4}\right|$

$=\sum_{r=1}^{2}\left|r^2-\frac{5}{4}\right|+\sum_{r=2}^{n}\left|r^2-\frac{5}{4}\right|=\frac{1}{4}+\sum_{r=2}^{n}r^2-\sum_{r=2}^{n}\frac{5}{4}$

46. $T_r=\sum T_r-\sum T_{r-1}=r^2+r$

$\sum_{r=1}^{n}\frac{2008}{T_r}=2008\sum_{r=1}^{n}\frac{1}{r(r+1)}=(2008)\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=(2008)\frac{n}{n+1}$

$\lim_{h\to\infty}\frac{(2008)n}{n+1}=2008$

48. $P(x)=\sum_{r=1}^{n}\left(x-\frac{1}{r}\right)\left(x-\frac{1}{r+1}\right)\left(x-\frac{1}{r+2}\right)$

Absolute term $=-\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=-\frac{1}{2}\left[\sum_{r=1}^{n}\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$

$$= -\frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$\lim_{n\to\infty} -\frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = -\frac{1}{4}$$

50. $\frac{1}{T_1}, \frac{1}{T_2}, \frac{1}{T_3}, \ldots\ldots, \frac{1}{T_k}$ are in A.P.

$$\frac{T_2}{T_6} = \frac{\frac{1}{a}+5d}{\frac{1}{a}+d} = 9 \;\Rightarrow\; d = -\frac{2}{a}$$

$$\frac{T_{10}}{T_4} = \frac{\frac{1}{a}+3d}{\frac{1}{a}+9d} = \frac{5}{17}$$

52. $\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\ldots\right)+\frac{2}{3}\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\ldots\right)=\frac{15}{8}$

53. $(x-1)(x-2)(x-3)(x-4)\ldots\ldots(x-10)$

Coefficient of x^8 = sum of terms taken two at a time

$$= \frac{1}{2}[(1+2+3+\ldots\ldots+10)^2 - (1^2+2^2+\ldots\ldots+10^2)]$$

55. AM = GM

$$\frac{\alpha+\beta+\gamma+\delta}{4} = (\alpha\beta\gamma\delta)^{1/4} = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = \beta = \gamma = \delta = \frac{1}{2}$$

56. Use AM ≥ GM

57. $\sum_{r=1}^{\infty}(\alpha^r+\beta^r) = (\alpha+\alpha^2+\alpha^3+\ldots)+(\beta+\beta^2+\beta^3+\ldots)$

$$= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta}$$

$4x^2+2x-1=0$ ⟨ roots α, β

$4\left(\frac{x}{1+x}\right)^2 + 2\left(\frac{x}{1+x}\right) - 1 = 0 \Rightarrow 5x^2 - 1 = 0$ ⟨ roots $\frac{\alpha}{1-\alpha}$, $\frac{\beta}{1-\beta}$

58. $2^2[1+2^3+3^3+4^3+...+10^3]=4\left[\frac{10\times 11}{2}\right]^2=12100$

59. AM $\geq$ HM

$$\frac{b+\frac{a}{2}+\frac{a}{2}}{3}\geq\frac{3}{\frac{4}{a}+\frac{1}{b}}$$

60. $4^x-15=4^{2-x}\Rightarrow 4^x=16\Rightarrow x=2$

Common ratio $=\cos\left(\frac{2011\pi}{3}\right)=\cos\left(670\pi+\frac{\pi}{3}\right)=\frac{1}{2}$

61. AM $\geq$ GM

$$\frac{a^4+b^4+\frac{c^2}{2}+\frac{c^2}{2}}{4}\geq\left(\frac{a^4b^4c^4}{4}\right)^{1/4}$$

62. $x^2+y^2=x^2+\frac{1}{x^2}\geq 2$

63. $\frac{2}{1^3}+\frac{6}{1^3+2^3}+\frac{12}{1^3+2^3+3^3}+\frac{20}{1^3+2^3+3^3+4^3}+...\infty$

$$=\frac{1\times 2}{1^3}+\frac{2\times 3}{1^3+2^3}+\frac{3\times 4}{1^3+2^3+3^3}+...\infty$$

$$=\lim_{n\to\infty}\sum_1^n\frac{n(n+1)}{1^3+2^3+...+n^3}=\lim_{n\to\infty}\sum_1^n\frac{n(n+1)}{\left(\frac{n(n+1)}{2}\right)^2}$$

$$=\lim_{n\to\infty}4\sum_1^n\frac{1}{n(n+1)}=4\lim_{n\to\infty}\sum_1^n\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=4\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+...+\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=4\lim_{n\to\infty}\frac{n}{n+1}=4$$

64. $\frac{1}{(k-1)}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)(n+2)...(n+k-1)}-\frac{1}{(n+2)(n+3)...(n+k)}\right)=\frac{1}{(k-1)}\left(\frac{1}{2\cdot 3\cdot 4...k}\right)$

65. $A-G=\frac{3}{2}$ and $G-H=\frac{6}{5}$

As we know,

$G^2=AH \quad\Rightarrow\quad G^2\left(\frac{3}{2}+G\right)\left(G-\frac{6}{5}\right)\quad\Rightarrow G=6$ and $A=\frac{15}{2}$

$$\Rightarrow \quad ab = 36 \text{ and } a + b = 15 \Rightarrow a = 12 \text{ and } b = 3$$

66. $S = \frac{2+5}{2^2 \cdot 5^2} + \frac{5+8}{5^2 \cdot 8^2} + \frac{8+11}{8^2 \cdot 11^2} + \ldots = \frac{1}{3}\left(\frac{5^2 - 2^2}{2^2 \cdot 5^2} + \frac{8^2 - 5^2}{5^2 \cdot 8^2} + \frac{11^2 - 8^2}{8^2 \cdot 11^2} + \ldots\right)$

$$= \frac{1}{3}\left(\frac{1}{2^2} - \frac{1}{5^2} + \frac{1}{5^2} - \frac{1}{8^2} + \frac{1}{8^2} - \frac{1}{11^2} + \ldots + \frac{1}{29^2} - \frac{1}{32^2}\right)$$

$$= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{32^2}\right) = \frac{85}{1024}$$

67. $\sum_{r=1}^{10} \frac{r}{(r^2-1)^2 - r^2} = \sum_{r=1}^{10} \frac{r}{(r^2 - r - 1)(r^2 + r - 1)}$

$$= \frac{1}{2}\sum_{r=1}^{10}\left(\frac{1}{r^2 - r - 1} - \frac{1}{r^2 + r - 1}\right)$$

68. $\sum_{r=1}^{\infty} t_r = \sum_{r=1}^{\infty} \frac{r}{r^4 + r^2 + 1}$

$$= \sum_{r=1}^{\infty} \frac{r}{(r^2+1)^2 - r^2} = \sum_{r=1}^{\infty} \frac{r}{(r^2 - r + 1)(r^2 + r + 1)}$$

$$= \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right)$$

69. $S_\infty = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots$

$$\frac{1}{5} \cdot S_\infty = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots$$

$$\Rightarrow \quad \frac{4}{5} S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots = \frac{7}{4}$$

$$\Rightarrow \quad S_\infty = \frac{35}{16}$$

71. $x_1, x_2, x_3 \ldots x_{2n}$

$\sum_{r=1}^{2n} (-1)^{r+1} x_r^2$

$x_1^2 - x_2^2 + x_3^2 + \ldots - x_{2n}^2$

$(x_1 - x_2)(x_1 + x_2 + x_3 + \ldots x_{2n})$

$-(x_2 - x_1)(x_1 + x_2 + x_3 + \ldots x_{2n})$

$-\frac{(x_{2n} - x_1)}{2n - 1} \frac{2x}{2}[x_1 + x_{2n}]$

$\frac{x}{2x-1}(x_1^2 - x_{2n}^2)$

72. $\frac{\alpha+\beta}{2}=9; \quad \sqrt{\alpha\beta}=4$

73. rms $\geq$ AM

$$\sqrt{\frac{p^2+q^2}{2}} \geq \frac{p+q}{2}$$

74. $150\times 9+\frac{n}{2}[300+(n-1)(-2)]=4500 \Rightarrow n=25$

Total term $=n+9=34$

75. $S_{20}=\frac{20}{2}[2(1-ad)+19d]=20$

$$\Rightarrow \qquad 19d-2ad=0$$

76. $\sum_{n=3}^{\infty}\frac{1}{(n-2)(n-1)n(n+1)(n+2)}=\frac{1}{4}\sum_{n=3}^{\infty}\left(\frac{1}{(n-2)(n-1)n(n+1)}-\frac{1}{(n-1)n(n+1)(n+2)}\right)$

78. $2^x+2^{2x+1}+\frac{5}{2^x}=2^x+2^{2x}+2^{2x}+\frac{1}{2^x}+\frac{1}{2^x}+\frac{1}{2^x}+\frac{1}{2^x}+\frac{1}{2^x}$

$$\Rightarrow \quad \frac{2^x+2^{2x+1}+(5/2^x)}{8}\geq\left(2^x\times(2^{2x})^2\times\frac{1}{(2^x)^5}\right)^{1/8}=1$$

$$\Rightarrow \quad 2^x+2^{2x+1}+\frac{5}{2^x}\geq 8$$

79. $\sum_{r=1}^{\infty}\left(\frac{(4r+5)}{r(5r+5)}\right)\cdot\frac{1}{5^r}=\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{5r+5}\right)\cdot\frac{1}{5^r}=\sum_{r=1}^{\infty}\left(\frac{1}{r\cdot 5^r}-\frac{1}{(r+1)\cdot 5^{r+1}}\right)=\frac{1}{5}$

Exercise-2 : One or More than One Answer is/are Correct

1. $a=\frac{a_1+a_n}{2}, \quad b=\sqrt{a_1a_n}, \quad c=\frac{2a_1a_n}{a_1+a_n}$

$\Rightarrow \quad a\geq b\geq c$ and $b^2=ac$

2. $D_1: b^2-4ac<0$

$D_2: c^2-4ab<0$

$D_3: a^2-4bc<0$

$D_1+D_2+D_3: a^2+b^2+c^2<4(ab+bc+ac)$

$$1<\frac{a^2+b^2+c^2}{ab+bc+ac}<4$$

3. If a, b, c are in H.P.

A.M. > H.M.

$$\Rightarrow \quad \frac{a+c}{2} > b \Rightarrow a+c > 2b$$

$$\Rightarrow \quad a-b > b-c$$

or $$\frac{1}{a-b} - \frac{1}{b-c} < 0$$

G.M. > H.M.

also $\sqrt{ac} > b$ or $ac > b^2$

4. $T_p = a+(p-1)d = \dfrac{1}{q(p+q)}$

$T_q = a+(q-1)d = \dfrac{1}{p(p+q)} \Rightarrow a = d = \dfrac{1}{pq(p+q)}$

5. (a) $a, H_1, H_2, H_3, \ldots\ldots, H_n, b$ are in H.P.

$\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \dfrac{1}{H_3}, \ldots\ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ are in A.P.

$$\Rightarrow \quad \frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots\ldots + \frac{1}{H_n} = \frac{n}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

(c) $a, A_1, A_2, A_3, \ldots\ldots, A_{2n}, b$ are in A.P.

$A_1 + A_{2n} = A_2 + A_{2n-1} = A_3 + A_{2n-2} = \ldots\ldots = a+b$

(d) $4g_2 + 5g_3 = 4r + 5r^2$

This is minimum at $r = -\dfrac{2}{5}$

6. a, b, c are in H.P.

$\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

(a) $\dfrac{a+b+c}{a} - 2, \dfrac{a+b+c}{b} - 2, \dfrac{a+b+c}{c} - 2$ are in A.P.

(b) $\dfrac{a+b+c}{c} - 1, \dfrac{a+b+c}{b} - 1, \dfrac{a+b+c}{c} - 1$ are in A.P.

(c) $\dfrac{2}{b} = \dfrac{1}{a} + \dfrac{1}{c} \geq \dfrac{2}{\sqrt{ac}} \Rightarrow \sqrt{ac} \geq b$

$a^5 + c^5 \geq 2\,(ac)^{5/2} \geq b^5$

(d) $2ac = ab + bc$

7. Let the roots be a, ar, ar^2, ar^3 and ar^4.

$$\frac{a(r^5-1)}{r-1} = 40 \qquad \ldots(1)$$

and $$\frac{1}{a}\frac{\left(\frac{1}{r^5}-1\right)}{\frac{1}{r}-1}=10 \quad ...(2)$$

put $\frac{r^5-1}{r-1}=\frac{40}{a}$ in (2) we get $ar^2=\pm 2$

Now, $\delta=(ar^2)^5=(\pm 2)^5$

8. (a) $\because$ $2a_{k+1}=a_k+a_{k+2}$

$\therefore$ $f_k(-1)=0$

-1 is a root.

$\therefore$ Other is also real root.

(b) From (a) (-1) is root for any 'k' so any pair of equation has a common root.

(c) If one root is -1, other roots are $-c/a$ (form)

$\frac{a_{k+2}}{a_k}$ i.e., $\frac{a_3}{a_1},\frac{a_4}{a_2},\frac{a_5}{a_3}$...... are not in A.P.

9. $b=\frac{a+c}{2}, d=\frac{2ce}{c+e}$

if $c^2=bd$, then $c^2=36$ $\quad (\because a=2, e=18)$

10. If a, b, c are in A.P. then

$$a=b-d \text{ and } c=b+d$$

$$a+b+c=60 \Rightarrow b=20$$

If $(a-2), b, (c+3)$ are in G.P., then

$$400=(18-d)(23+d) \Rightarrow d=2,-7$$

12. $\frac{81+144a^4+16b^4+9c^4}{4}\geq 36abc$

$\Rightarrow$ A.M. = G.M.

$\Rightarrow$ $81=144a^4=16b^4=9c^4$

13. x, y, z A.P.

Let $x=y-\theta$ and $z=y+\theta$

$$\cos(y-\theta)+\cos y+\cos(y+\theta)=1=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\cos(y)$$

$\sin(y-\theta)+\sin y+\sin(y+\theta)=\frac{1}{\sqrt{2}}=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\sin(y)\Rightarrow\cot y=\sqrt{2}$

$\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}=\sqrt{\frac{3}{2}}=3-4\sin^2\frac{\theta}{2}\Rightarrow\cos\theta=\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}}$

15. $\frac{10^{n+1}+1}{10^{n+2}+1}<\frac{10^{m+1}+1}{10^{m+2}+1}$

$\Rightarrow\ 10^{n+1}\cdot10^{m+2}+10^{n+1}+10^{m+2}+1<10^{n+2}\cdot10^{m+1}+10^{n+2}+10^{m+1}+1$

$\Rightarrow\ 10^{m+1}<10^{n+1}$

16. $S_r=\sqrt{r+S_r}\ \Rightarrow\ S_r^2-S_r=r$

17. 50, 48, 46, 44, ... A.P.

$T_n=50+(n-1)(-2)=0$

$\Rightarrow\qquad n=26$

18. $S_n=S_n\sum_{r=1}^{n}t_r=\sum_{r=1}^{n}\frac{2r+1}{1^2+2^2+3^2+\ldots+r^2}=\sum_{r=1}^{n}6\left(\frac{1}{r}-\frac{1}{r+1}\right)=6\left(1-\frac{1}{n+1}\right)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

Sol. $T_1=A+B=0\qquad\Rightarrow A=-B$

$T_2=A\alpha+B\beta=1\qquad\Rightarrow A(\alpha-\beta)=1$

$T_3=A\alpha^2+B\beta^2=1\qquad\Rightarrow A(\alpha^2-\beta^2)=1$

$T_4=A\alpha^3+B\beta^3=2\qquad\Rightarrow A(\alpha^3-\beta^3)=2$

$\Rightarrow\ \alpha+\beta=1$ and $\alpha\beta=-1$

Paragraph for Question Nos. 3 to 4

Sol. Set A: $5-D, 5, 5+D$ and

Set B: $5-d, 5, 5+d$

$\frac{p}{q}=\frac{25-D^2}{25-d^2}=\frac{7}{8}$

$\Rightarrow\qquad 25=8D^2-7d^2=d^2+16d+8\qquad(\because D=1+d)$

$\Rightarrow\qquad d=1$ and $D=2$

Set A {3, 5, 7} and set B {4, 5, 6}

Paragraph for Question Nos. 5 to 7

5. $\frac{(x-3)+(y+1)+(z+5)}{3} \geq [(x-3)(y+1)(z+5)]^{1/3}$

$\Rightarrow \quad (x-3)(y+1)(z+5) \leq (21)^3$

6. Term is $6(x-3)\left(y+\frac{1}{2}\right)\left(z+\frac{5}{3}\right) \Rightarrow \frac{(x-3)+y+\frac{1}{2}+z+\frac{5}{3}}{3} \geq \left[(x-3)\left(y+\frac{1}{2}\right)\left(z+\frac{5}{3}\right)\right]^{1/3}$

$(x-3)\left(y+\frac{1}{2}\right)\left(z+\frac{5}{3}\right) \leq \frac{(355)^3}{6^3 \times 3^3}$

Maximum value $= \frac{(355)^3}{6^2 \times 3^3}$

7. $\frac{x+y+z}{3} \geq (xyz)^{1/3}$; $\quad xyz \leq (20)^3$

Paragraph for Question Nos. 8 to 10

Sol. Let removed number are A and $A+1$.

$$\frac{n(n+1)}{2} - 2A - 1 = (n-2)\frac{105}{4}$$

$$2n^2 - 103n + 206 = 8A$$

$$\Rightarrow \quad n = 50, A = 7$$

Paragraph for Question Nos. 11 to 13

Sol. $a_{n+1} - 1 = (a_n - 1)^2$

$a_n - 1 = (a_{n-1} - 1)^2$

$a_{n-1} - 1 = (a_{n-2} - 1)^2$

$(a_2 - 1) = (a_1 - 1)^2$

$a_1 - 1 = (a_0 - 1)^2$

$\Rightarrow \quad (a_n - 1)(a_{n-1} - 1)^2(a_{n-2} - 1)^{2^2} \ldots\ldots (a_1 - 1)^{2^{n-1}} = (a_{n-1} - 1)^2 (a_{n-2} - 1)^2 \ldots\ldots (a_0 - 1)^{2^n}$

$\Rightarrow \quad (a_n - 1) = 3^{2^n}$

$$b_n = \frac{2 \cdot (3^{2^0} + 1)(3^2 + 1) \ldots\ldots (3^{2^{n-1}} + 1)}{(3^{2n} + 1)}$$

$$b_n = \frac{3^{2^n} - 1}{3^{2^n} + 1}$$

Paragraph for Question Nos. 14 to 15

$$f(n)=\sum_{r=2}^{n}\frac{4}{(r-1)r(r+1)}=2\sum_{r=2}^{n}\left(\frac{1}{(r-1)r}-\frac{1}{r(r+1)}\right)=2\left(\frac{1}{1\cdot 2}-\frac{1}{n(n+1)}\right);\ a=\lim_{n\to\infty}f(n)=1$$

14. $f(7)+f(8)=\frac{122}{63}$

15. $x^2+\frac{3}{2}x+t=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix}$

Paragraph for Question Nos. 16 to 17

Sol. $\frac{a_1}{a_1+1}=\frac{a_2}{a_2+3}=\frac{a_3}{a_3+5}=\ldots\ldots\frac{a_{1005}}{a_{1005}+2009}=\frac{1}{k}$

$a_1=\frac{1}{k-1}, a_2=\frac{3}{k-1}, a_3=\frac{5}{k-1},\ldots\ldots a_{1005}=\frac{2009}{k-1}$

$a_1+a_2+a_3+\ldots\ldots+a_{1005}=\frac{(1005)^2}{k-1}=2010 \Rightarrow k-1=\frac{1005}{2}$

Exercise-4 : Matching Type Problems

1. (A) a, b, c are in A.P.

$b-a=c-b$

$b-a, c-b, a$ are in G.P.

$\frac{c-b}{b-a}=\frac{a}{c-b} \Rightarrow c-b=a \quad (\because b-a=c-b)$

(B) a, x, b are in A.P.

$x=\frac{a+b}{2}$

a, y, z, b are in G.P.

$y=a^{2/3}b^{1/3},\quad z=a^{1/3}b^{2/3}$

(C) $a, b=ar, c=ar^2$

If $c>4b-3a$

$r^2-4r+3>0 \qquad (\because a>0)$

$(r-3)(r-1)>0$

(D) $7x^2-8x+9<0$

$a=7>0,\quad D=64-252<0$

No solution

2. (A) $a+d=b+c=20$

(B) $2, G_1, G_2, G_3, G_4, G_5, G_6, 5$ are in G.P.

$G_1G_6 = G_2G_5 = G_3G_4 = 10$

(C) $a_4h_7 = a_1h_{10} = a_{10}h_1 = 6$

(D) $(2^x-5)^2 = 2\left(2^x - \frac{7}{2}\right) \Rightarrow (2^x-8)(2^x-4)=0 \Rightarrow x=3$

3. (A) $2 \cdot 2^{x^2} = 2^x + 2^{x^3}$

Exponential series can't be in A.P.

(B) If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in A.P.

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots\ldots = a_n - a_{n-1} = d$$

$$S = -d\left(\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}\right)$$

$$= -d\left(\frac{\sqrt{a_n}-\sqrt{a_1}}{d}\right) = \sqrt{a_1} - \sqrt{a_n}$$

(C)
$$\frac{S_{2n}}{S_n} = \frac{\frac{2n}{2}[2a+(2n-1)d]}{\frac{n}{2}[2a+(n-1)d]} = 3$$

$$\Rightarrow \quad 2a = (n+1)d$$

$$\frac{S_{3n}}{2S_n} = \frac{\frac{3n}{2}[2a+(3n-1)d]}{\frac{2n}{2}[2a+(n-1)d]} = 3$$

(D) $t_1 + t_5 = t_2 + t_4 = 2t_3$

$$\frac{4(t_1 - t_2 - t_4) + 6t_3 + t_5}{3t_1} = \frac{3t_1 + (t_1+t_5) - 4(t_2+t_4) + 3(2t_3)}{3t_1} = 1$$

4. A → Q; B → P; C → T; D → S

5. (A) $\frac{1}{3}\log_2 x + \log_2 y = 5$ and $\frac{1}{3}\log_2 y + \log_2 x = 7$

$\Rightarrow \quad \log_2 x = 6$ and $\log_2 y = 3$

$\Rightarrow \quad x = 2^6$ and $y = 2^3$

(B) $\angle B = 60°$ and $b^2 = ac$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{1}{2} \quad \Rightarrow a^2 + c^2 = 2ac$$

$$\Rightarrow \quad a = c$$

(C) AM ≥ GM

$$\frac{\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b}+\frac{a}{c}+\frac{b}{c}}{6} \geq 1$$

(D) $(b+c)^2 - a^2 = \lambda bc$

$\Rightarrow \quad b^2 + c^2 - a^2 = (\lambda - 2)\, bc$

$\Rightarrow \quad \frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$

$-1 < \frac{\lambda - 2}{2} < 1$

$0 < \lambda < 4$

6. $P(n) \cdot (f(n+2) - f(n)) = q(n)$

$P(n) \cdot \left(\frac{1}{n+1} + \frac{1}{n+2}\right) = q(n)$

$P(n) \cdot (2n+3) = (n^2 + 3n + 2) \cdot q(n)$

$\Rightarrow \quad P(n) = n^2 + 3n + 2$ and $q(n) = (2n+3)$

Exercise-5 : Subjective Type Problems

1. If a, b, c, d are in A.P. with common difference 'k', then

$9k^3 + (x-4)k^2 + 4k = 0$

$k\{9k^2 + (x-4)k + 4\} = 0$

$D \geq 0 \quad \Rightarrow \quad (x-4)^2 - 144 \geq 0$

$(x+8)(x-16) \geq 0$

$\Rightarrow \quad x \in (-\infty, -8] \cup [16, \infty)$

2. $S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots\ldots + n \cdot 2^n$

$2 \cdot S = 1 \cdot 2^2 + 2 \cdot 2^3 + 3 \cdot 2^4 + \ldots\ldots + (n-1) \cdot 2^n + n \cdot 2^{n+1}$

$\Rightarrow \quad S = (n-1) \cdot 2^{n+1} + 2 = 2 + 2^{n+10}$

$\Rightarrow \quad 2(n-1) = 2^{10}$

$\Rightarrow \quad n = 513$

3. $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r+2}{2^{r+1} r(r+1)} = \sum_{r=1}^{\infty} \left[\frac{1}{r \cdot 2^r} - \frac{1}{(r+1) \cdot 2^{r+1}}\right] = \frac{1}{2}$

4. $\sum_{r=1}^{\infty} \frac{8r}{4r^4 + 1} = 2 \sum_{r=1}^{\infty} \left(\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right) = 2$

5. Let three terms in A.P. $a-d, a, a+d$

If $(a-d)^2, a^2, (a+d)^2$ are in G.P. $\Rightarrow d = \pm\sqrt{2}\,a$

$$r = \frac{a^2}{(a-d)^2} = \frac{1}{(1\pm\sqrt{2})^2}$$

6. $\sqrt{\frac{10^{2n}-1}{9} - 2\left(\frac{10^n-1}{9}\right)} = P\left(\frac{10^n-1}{9}\right) \Rightarrow P = 3$

7. $a-d, a, a+d, a-d+30$

If last three terms are in G.P.

$$(a+d)^2 = a(a-d+30)$$

$$\Rightarrow \qquad a = \frac{d^2}{30-3d}$$

8. $\frac{1}{8n^4}\sum_{k=1}^{n}[k(k+2)(k+4)(k+6)-(k-2)k(k+2)(k+4)]$

$$\frac{1}{8}\left[\frac{(n-1)(n+1)(n+3)(n+5)+n(n+2)(n+4)(n+6)+15}{n^4}\right] = \frac{1}{4}(n\to\infty)$$

9. Unit digit of $\left[\frac{n(n+1)}{2}\right]^2 = 1$

Then unit digit of $\frac{n(n+1)}{2}$ is 1 because unit digit of $n(n+1)$ can not be 8.

10. $2\log_b c = \log_c a + \log_a b$

$$2\left(\frac{\log a + 2\log r}{\log a + \log r}\right) = \left(\frac{\log a}{\log a + 2\log r}\right) + \left(\frac{\log a + \log r}{\log a}\right)$$

Let $A = \log a$ and $R = \log r \Rightarrow 3A^2 + 3Ar - 2R^2 = 0 \Rightarrow \frac{A}{R} = \frac{-3+\sqrt{33}}{6}$

$$d = \log_b c - \log_c a = \frac{A+2R}{A+R} - \frac{A}{A+2R} = \frac{3}{2}$$

11. $3, \frac{3}{r}, \frac{3r}{s}, 7s;\quad \frac{2}{r} = 1 + \frac{r}{s}$ and $\frac{6r}{s} = \frac{3}{r} + 7s$

$\Rightarrow\quad 7r^3 - 6r^2 + 21r - 18 = 0 \Rightarrow (r^2+3)(7r-6) = 0$

$$\Rightarrow \qquad r = \frac{6}{7} \text{ and } s = \frac{9}{14}$$

12. $$S = \frac{1^2}{3^1} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \frac{4^2}{3^4} + \ldots\ldots$$

$$\frac{S}{3} = \frac{1^2}{3^2} + \frac{2^2}{3^3} + \frac{3^2}{3^4} + \ldots\ldots$$

$$\frac{2S}{3}=S-\frac{S}{3}=\frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\frac{7}{3^4}+\ldots\ldots$$

$$\frac{2S}{9}=\frac{1}{3^2}+\frac{3}{3^3}+\frac{5}{3^4}+\ldots\ldots$$

$$\frac{2S}{3}-\frac{2S}{9}=\frac{1}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\frac{2}{3^4}+\ldots\ldots$$

$$\frac{4S}{9}=\frac{1}{3}+\frac{2}{3^2}\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\right)$$

$$\frac{4S}{9}=\frac{1}{3}+\frac{2}{9}\left(\frac{1}{2/3}\right)=\frac{2}{3} \Rightarrow S=\frac{3}{2}=\frac{p}{q}$$

13. $S_\infty = f(x)_{\max} \qquad x\in[-4,3]$

$a-ar=f'(0)=3$

$f'(x)=3x^2+3>0 \qquad \therefore\ f(x)_{\max}=f(3)=27+9-9=27$

$$S_\infty=27=\frac{a}{1-r}$$

$$a(1-r)=3 \Rightarrow \frac{1}{1-r}=\frac{a}{3}$$

$$\therefore\quad 27=a\left(\frac{a}{3}\right)$$

$$a^2=81 \Rightarrow a=\pm 9$$

If $a=9 \qquad 1-r=\frac{3}{9}, \quad r=\frac{2}{3}$ If $a=-9 \qquad 1-r=-\frac{1}{3}, \quad r=\frac{4}{3}>1$ (rejected)

$$\therefore\quad \frac{p}{q}=\frac{2}{3} \qquad \therefore\ p+q=5$$

14. Total runs from 1 to 9 = 1350

Let, number of terms in A.P. be n.

$\Rightarrow \quad \frac{n}{2}[300+(n-1)\times(-1)]=4500-1350=3150$

$\Rightarrow \quad n=25$ or 126, $n=126$ (not possible)

$\Rightarrow \quad n=25$, total matches = 34

15. $x=\frac{10}{4}\sum_{n=3}^{100}\left(\frac{1}{n-2}-\frac{1}{n+2}\right)=\frac{10}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{102}-\frac{1}{101}-\frac{1}{100}-\frac{1}{99}\right)$

16. $$f(n)=\frac{(2n+1)+(2n-1)+\sqrt{(2n+1)(2n-1)}}{\sqrt{2n+1}+\sqrt{2n-1}}$$

Let $\sqrt{2n+1}=a$ and $\sqrt{2n-1}=b$

$$f(n)=\frac{(a^2+b^2+ab)}{(a+b)}\frac{(a-b)}{(a-b)}=\frac{a^3-b^3}{a^2-b^2}$$

$$\Rightarrow \quad f(n)=\frac{(2n+1)^{3/2}-(2n-1)^{3/2}}{2}$$

$$\sum_{n=1}^{60} f(n)=\sum_{n=1}^{60}\frac{(2n+1)^{3/2}-(2n-1)^{3/2}}{2}=\frac{(121)^{3/2}-1}{2}=665$$

17. $3^0\{2^0+2^{-1}+2^{-2}\ldots\infty\}=1\{2\}$

$3^{-1}\{2^0+2^{-1}+2^{-2}\ldots\infty\}=\frac{1}{3}\{2\}$

$3^{-2}\{2^0+2^{-1}+2^{-2}\ldots\infty\}=\frac{1}{3}\{2\}$

$\vdots \qquad\qquad \vdots$

$\qquad\qquad \infty$

Hence, $\dfrac{2\times 1}{1-\frac{1}{3}}=3$

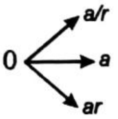
18. $15^2+(15+d)^2+(15+2d)^2+\ldots+(15+9d)^2=1185$

$\Rightarrow \quad 19d^2+90d+71=0$

$\Rightarrow \quad d=-1$

$S_n \ge S_{n-1}$

$\frac{n}{2}(31-n)\ge\left(\frac{n-1}{2}\right)(32-n) \Rightarrow n\le 16$

19. $24x^3-14x^2+kx+3=0$ 

Product of roots $a^3=-\frac{1}{8} \Rightarrow a=-\frac{1}{2}$

$\Rightarrow \quad k=-7$

If $x=7$ lies between the roots, then

$$f(7)=49+7\alpha^2-112<0$$
$$\alpha^2-9<0$$

20. $9x^3+3y^3+1=9xy$

$(9^{1/3}x)^3+(3^{1/3}y)^3+1^3=3(9^{1/3}x)(3^{1/3}y) \Rightarrow 9^{1/3}x=3^{1/3}y=1$

21. If a, x, y, z, b A.P.

$x = \dfrac{3a+b}{4}, y = \dfrac{a+b}{2}$ and $z = \dfrac{a+3b}{4}$

If a, x, y, z, b H.P.

$x = \dfrac{4ab}{3b+a}, y = \dfrac{2ab}{a+b}$ and $z = \dfrac{4ab}{3a+b}$

If $\left(\dfrac{3a+b}{4}\right)\left(\dfrac{a+b}{2}\right)\left(\dfrac{a+3b}{4}\right) = 55$ and $\left(\dfrac{4ab}{3b+a}\right)\left(\dfrac{2ab}{a+b}\right)\left(\dfrac{4ab}{3a+b}\right) = \dfrac{343}{55} \Rightarrow ab = 7$

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Chapter 10 – Determinants

10 DETERMINANTS

Exercise-1 : Single Choice Problems

1. Direct expansion.

2. $$D=\begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}=0 \qquad \Rightarrow (k-1)^2(k+2)=0$$

also $$D_1=\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ k^2 & 1 & k \end{vmatrix}\neq 0 \qquad \Rightarrow k\neq 1 \Rightarrow k=-2$$

3. $$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=(1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$$

4. $$D=\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix}=0 \quad \Rightarrow \frac{2}{b}=\frac{1}{a}+\frac{1}{c}$$

6. $$2x+ay+6z=8$$

and $4x+2ay+6z=8 \quad \Rightarrow \quad 2x+ay=0$

and $6x+12y+6z=30 \quad \Rightarrow \quad 4x+(12-a)\,y=22$

$\Rightarrow \qquad y=\dfrac{22}{12-3a} \qquad a\neq 4$

7. $R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix} x^2-4 & x^2-4 & x^2-4 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix}=(x^2-4)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix}=0$$

$\Rightarrow \quad (x^2-4)(x^2-15)(20-x^2)=0$

8. $D=\begin{vmatrix} k & k+1 & k-1 \\ k+1 & k & k+2 \\ k-1 & k+2 & k \end{vmatrix}=0$

$R_1 \to R_1 - R_2$ $\qquad$ $R_2 \to R_2 - R_3$

$D=\begin{vmatrix} -1 & 1 & -3 \\ 2 & -2 & 2 \\ k-1 & k+2 & k \end{vmatrix}=0$

9. $\Delta=\begin{vmatrix} \log a+(n-1)\log r & \log a+(n+1)\log r & \log a+(n+3)\log r \\ \log a+(n+5)\log r & \log a+(n+7)\log r & \log a+(n+9)\log r \\ \log a+(n+11)\log r & \log a+(n+13)\log r & \log a+(n+15)\log r \end{vmatrix}$

$C_3 \to C_3 - C_2$ $\qquad$ $C_2 \to C_2 - C_1$

$\Rightarrow \begin{vmatrix} \log a+(n-1)\log r & 2\log r & 2\log r \\ \log a+(n+5)\log r & 2\log r & 2\log r \\ \log a+(n+11)\log r & 2\log r & 2\log r \end{vmatrix}=0$

10. $D_2=\begin{vmatrix} a_1 & 2a_3 & 5a_2 \\ b_1 & 2b_3 & 5b_2 \\ c_1 & 2c_3 & 5c_2 \end{vmatrix}=10\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix}=-10\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

11. $\Delta_2=\begin{vmatrix} 1 & bc & a \\ 1 & ac & b \\ 1 & ab & c \end{vmatrix}$

$R_1 \to aR_1 \qquad R_2 \to bR_2 \qquad R_3 \to cR_3$

$\Delta_2=\frac{1}{abc}\begin{vmatrix} a & abc & a^2 \\ b & abc & b^2 \\ c & abc & c^2 \end{vmatrix}=\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}=-\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=-\Delta_1$

12. $C_1 \to C_1 - C_2 + C_3 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & 1-a \\ 1 & a & 1+a-b \end{vmatrix}=1$

13. $\begin{vmatrix} 1 & 2 & x \\ 2 & 3 & x^2 \\ 3 & 5 & 2 \end{vmatrix}=10 \Rightarrow x^2+x-12=0$

Sum $=-1$

14. $R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$D=\begin{vmatrix} -1 & -1 & -1 \\ d-a+1 & e-b+1 & f-c+1 \\ x+a & x+b & x+c \end{vmatrix},\quad C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3$$

On solving D does not depend on x.

15. $R_1 \to R_1 + R_2 + R_3$

$$\Delta=(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}\quad C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3$$

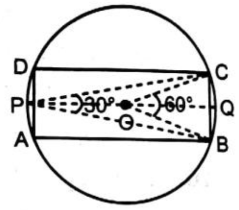
$$\Delta=(x+y+z)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2y \\ 0 & 1 & z-x-y \end{vmatrix} \Rightarrow \Delta=(x+y+z)^3$$

16. $\angle BOC = 60^\circ$

$\Rightarrow \quad BC = OB = OC = r$

$AB = 2r\cos 30^\circ = \sqrt{3}\, r$

$$\frac{\text{Area or rectangle}}{\text{Area of circle}} = \frac{\sqrt{3}\, r^2}{\pi r^2} = \frac{\sqrt{3}}{\pi}$$



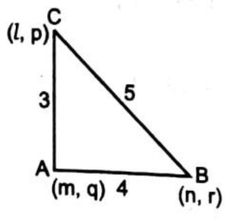
17. $C_1 \to C_1 - bC_3,\ C_2 \to C_2 + aC_3$

$$(1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}=(1+a^2+b^2)^3$$

18. $$\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ c+d & a+b & 0 \\ cd & ab & 0 \end{vmatrix}\begin{vmatrix} 1 & a+b & ab \\ 1 & c+d & cd \\ 0 & 0 & 0 \end{vmatrix}$$

$$=0$$

19. $$|B| = \begin{vmatrix} l & m & n \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = [2\text{Ar}(\Delta ABC)]^2$$



20. $$D=\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 10 \end{vmatrix}=0$$

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ K & 3 & 4 \\ K^2 & 5 & 10 \end{vmatrix} = 5(K^2 - 3K + 2) = 5(K-1)(K-2)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & K & 4 \\ 1 & K^2 & 10 \end{vmatrix} = -3(K^2 - 3K + 2) = -3(K-2)(K-1)$$

$$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & K \\ 1 & 5 & K^2 \end{vmatrix} = K^2 - 3K + 2 = (K-2)(K-1)$$

21. $(x+1)(x+2)(x+3)\begin{vmatrix} 1 & x+1 & (x+1)^2 \\ 1 & x+2 & (x+2)^2 \\ 1 & x+3 & (x+3)^2 \end{vmatrix} = 2(x+1)(x+2)(x+3)$

22. $\begin{vmatrix} -2 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

$-2(1-\cos^2 A) - \cos C(-\cos C - \cos A \cos B) + \cos B(\cos C \cos A + \cos B)$

$-2 + \cos 2A + \dfrac{1+\cos 2C}{2} + \dfrac{1+\cos 2B}{2} + 2\cos A \cos B \cos C$

$\cos 2A + \cos 2C + \cos 2B + 2\cos A \cos B \cos C$

$2\cos(A+B)\cos(A+B) + 2\cos^2 C - 1 + 2\cos A \cos B \cos C$

$2\cos C[\cos C - \cos(A-B)]$

$-2\cos C \cos A \cos B - 1 + 2\cos A \cos B \cos C = -1$

24. As a, b and c are the roots of $x^3 + 2x^2 + 1 = 0$, we have

$$a + b + c = -2$$
$$ab + bc + ca = 0$$
$$abc = -1$$

Now, for finding the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, evaluating using first row, we get

$$a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) = abc - a^3 - b^3 + abc + abc - c^3$$
$$= 3abc - a^3 - b^3 - c^3 = -(a^3 + b^3 + c^3 - 3abc)$$
$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= -(-2)[(-2)^2 - 3(0)] = 8$$

25. For non-trivial solution, $|A|$ or $D = 0$, That is, $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ \lambda+1 & \lambda & \lambda+2 \\ \lambda-1 & \lambda+2 & \lambda \end{vmatrix} = 0$

Now, $R_2 \to R_2 - R_1; R_3 \to R_3 - R_1$ gives $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ 1 & -1 & 3 \\ -1 & 1 & 1 \end{vmatrix} = 0$

Also, $R_3 \to R_3 + R_2$ gives $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 0$

Evaluation using third row, we get

$$4(-\lambda - \lambda - 1) = 0 \Rightarrow \lambda = -\frac{1}{2}$$

which is exactly the real value of λ.

Exercise-2 : One or More than One Answer is/are Correct

1. $f(a, b) = a(a+b)(a+2b)$

2. $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \cos^2\theta & \sin^2\theta & 1+2\sqrt{3}\tan\theta \end{vmatrix} = 0 \Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}$$

3. $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Delta = d^2 \begin{vmatrix} -1 & -1 & 3 \\ -1 & 2 & -1 \\ a+2d & a & a+d \end{vmatrix} = -d^2(13d + 12a)$$

4. $\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$

5. $D(x) = \begin{vmatrix} x^2+4x-3 & 2x+4 & 13 \\ 2x^2+5x-9 & 4x+5 & 26 \\ 8x^2-6x+1 & 16x-6 & 104 \end{vmatrix}$

$C_3 \to C_3 - 4C_2, C_2 \to C_2 - 2C_1$

$$D(x) = \begin{vmatrix} 3x+3 & 3 & 0 \\ 26x-37 & 26 & 0 \\ 8x^2-6x+1 & 16x-6 & 104 \end{vmatrix}$$

7. $D=\begin{vmatrix} a & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & a \end{vmatrix}=0 \Rightarrow a^2-a-2=0 \Rightarrow (a-2)(a+1)=0$

$D_1=\begin{vmatrix} 0 & 1 & 2 \\ b & 2 & 1 \\ 0 & 1 & a \end{vmatrix}$ $D_2=\begin{vmatrix} a & 0 & 2 \\ 1 & b & 1 \\ 2 & 0 & a \end{vmatrix}$ $D_3=\begin{vmatrix} a & 1 & 0 \\ 1 & 2 & b \\ 2 & 1 & 0 \end{vmatrix}$

$a=2$ infinite solution

$a=-1, b\neq 0$ has no solution.

8. $D=\begin{vmatrix} a & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & a \end{vmatrix}=0 \quad \Rightarrow a^2-a-2=0$

$\Rightarrow (a-2)(a+1)=0$

$D=\begin{vmatrix} 0 & 1 & 2 \\ b & 2 & 1 \\ 0 & 1 & a \end{vmatrix}$ $D=\begin{vmatrix} a & 0 & 2 \\ 1 & b & 1 \\ 2 & 0 & a \end{vmatrix}$ $D=\begin{vmatrix} a & 1 & 0 \\ 1 & 2 & b \\ 2 & 1 & 0 \end{vmatrix}$

$a=2$ infinite solution

$a=-1, b\neq 0$ has no solution.

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

$D=\begin{vmatrix} 2 & \lambda & 6 \\ 1 & 2 & \mu \\ 1 & 1 & 3 \end{vmatrix}=(\lambda-2)(\mu-3);$ $D_1=\begin{vmatrix} 8 & \lambda & 6 \\ 5 & 2 & \mu \\ 4 & 1 & 3 \end{vmatrix}=(\lambda-2)(4\mu-15)$

$D_2=\begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & \mu \\ 1 & 4 & 3 \end{vmatrix}=0;$ $D_3=\begin{vmatrix} 2 & \lambda & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}=(\lambda-2)$

Exercise-4 : Subjective Type Problems

2. $R_1 \to R_1+R_2+R_3$

$3\begin{vmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \\ 2b_1+c_1 & 2b_2+c_2 & 2b_3+c_3 \\ 2c_1+a_1 & 2c_2+a_2 & 2c_3+a_3 \end{vmatrix}$

$$R_2 \to R_2 + R_1 - R_3 = 9\begin{vmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \\ b_1 & b_2 & b_3 \\ 2c_1+a_1 & 2c_2+a_2 & 2c_3+a_3 \end{vmatrix}$$

Now, operate as $R_3 \to R_3 - R_1 + R_2$

then $R_1 \to R_1 - R_2 - R_3$

3. Let $f(x) = \begin{vmatrix} (1+x)^2 & (1+x)^4 & (1+x)^6 \\ (1+x)^3 & (1+x)^6 & (1+x)^9 \\ (1+x)^4 & (1+x)^8 & (1+x)^{12} \end{vmatrix}$

Coefficient of 'x' is $f'(0)$.

$$f'(x) = \begin{vmatrix} 2(1+x)^2 & 4(1+x)^3 & 6(1+x)^5 \\ (1+x)^3 & (1+x)^6 & (1+x)^9 \\ (1+x)^4 & (1+x)^8 & (1+x)^{12} \end{vmatrix} + \begin{vmatrix} (1+x)^2 & (1+x)^4 & (1+x)^6 \\ 3(1+x)^2 & 6(1+x)^5 & 9(1+x)^8 \\ (1+x)^4 & (1+x)^8 & (1+x)^{12} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^2 & (1+x)^2 & (1+x)^6 \\ (1+x)^3 & (1+x)^6 & (1+x)^9 \\ 4(1+x)^3 & 8(1+x)^7 & 12(1+x)^{11} \end{vmatrix}$$

Put $x=0$, $f'(0)=0$

5. For non-zero solution, $\Delta = 0$

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & 2 & k \\ 4 & 1 & 1 \end{vmatrix} = 0 \Rightarrow k = 0$$

Now, let $x = \lambda$

So, $y = -\frac{3\lambda}{2}$, $z = -\frac{5\lambda}{2}$

$\Rightarrow$ Minimum positive integer value of z is at $\lambda = -2$; $z = 5$

6. $\begin{vmatrix} 2a & -2 & 3 \\ 1 & a & 2 \\ 2 & 0 & a \end{vmatrix} = 0 \Rightarrow a = 2$

7. Let three terms be $A-d, A, A+d$.

$\Rightarrow$ $A^4 = (A-d)^2(A+d)^2 = A^4 + d^4 - 2A^2d^2$

$\Rightarrow$ $d = \pm\sqrt{2}A, r = 3+2\sqrt{2}$ or $r = 3-2\sqrt{2}$

8. $\Delta_3 = \begin{vmatrix} 3a_1+b_1 & 3a_2+b_2 & 3a_3+b_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} = \begin{vmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} = 27\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\Delta_2 = \begin{vmatrix} 6a_1 & 2a_2 & 2a_3 \\ 3b_1 & b_2 & b_3 \\ 12c_1 & 4c_2 & 4c_3 \end{vmatrix} = 24\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

9. $\Delta = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 2 \end{vmatrix} = 3(1+\cos^2\theta)$

Its minimum value $= 3$

10. $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & \lambda \end{vmatrix} = \lambda - 8 = 0 \Rightarrow \lambda = 8$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 2 & 5 & \mu \end{vmatrix} = \mu - 36 = 0 \Rightarrow \mu = 36$$

11. $n\sin 2\pi\left(1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}\cdots\frac{1}{\lfloor N}\right)\lfloor n$

$n\sin 2\pi\left(1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots\frac{1}{(n+1)(n+2)\ldots(N)}\right)$

Using $\sin(2n\pi+\theta) = \sin\theta$

$$= n(2\pi)\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots\frac{1}{(n+1)(n+2)\ldots N}\right)$$

Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

$$= 2\pi$$

12. $\begin{vmatrix} \cos\theta & \sin\theta & \cos\theta \\ \sin\theta & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & -\cos\theta \end{vmatrix} = 0 \Rightarrow -2\cos\theta\cos 2\theta = 0$

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Chapter 11 – Complex Numbers

11 COMPLEX NUMBERS

Exercise-1 : Single Choice Problems

2. $\arg(z-2-7i)=\cot^{-1}(2) \Rightarrow \dfrac{y-7}{x-2}=\dfrac{1}{2}$

$\arg\left(\dfrac{z-5i}{z+2-i}\right)=\pm\dfrac{\pi}{2} \qquad \Rightarrow x(x+2)+(y-5)(y-1)=0$

4. $z_1^2+z_2^2=z_1z_2$

5. Let $\omega=re^{i\theta}$ then $z=\dfrac{1}{r}e^{i(\pi/2+\theta)}$

$$\bar{z}\omega=\frac{1}{r}e^{-i(\pi/2+\theta)}r\cdot e^{i\theta}=e^{-i\pi/2}$$

6. $a\sum_{r=1}^{n}r\omega^{r-1}+b\sum_{r=1}^{n}\omega^{r-1}=a(1+2\omega+3\omega^2+\ldots\ldots+n\omega^{n-1})+b(1+\omega+\omega^2+\ldots\ldots+\omega^{n-1})$

$$=a\left\{\frac{1+\omega+\omega^2+\ldots\ldots+\omega^{n-1}}{1-\omega}-\frac{n\omega^n}{1-\omega}\right\}+b(0)$$

$$=a\left(0-\frac{n}{1-\omega}\right)+0$$

8. $z^4+z^3+2=0$ has roots z_1,z_2,z_3 and z_4.

$\Rightarrow\ (z-1)^4+2(z-1)^3+32=0$ has roots $(2z_1+1),(2z_2+1),(2z_3+1)$ and $(2z_4+1)$

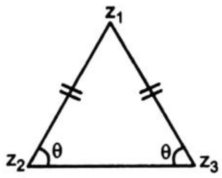
9. $\arg\left(\dfrac{z-6-3i}{z-3-6i}\right)=\dfrac{\pi}{4}$

$\Rightarrow\ (x-6)^2+(y-6)^2=9$

11. $|iz+z_1|=|i||z-iz_1|=|z-iz_1|$

Maximum distance of $iz_1(-3+5i)$ from z is $2+\sqrt{3^2+(5-1)^2}=7$

12.



$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{|z_1 - z_2|}{|z_3 - z_2|} e^{i\theta}$$

$$\frac{z_1 - z_3}{z_2 - z_3} = \frac{|z_1 - z_3|}{|z_2 - z_3|} e^{-i\theta}$$

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_2} + \frac{z_1 - z_3}{z_3 - z_2}\right) = \arg\left(\left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\theta} - \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{-i\theta}\right)$$

$$= \pm\frac{\pi}{2}$$

13. $\frac{z_2}{z_1} = \frac{3}{2} e^{i\pi/3}$

$$\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = \left|\frac{1 + \frac{3}{2} e^{i\pi/3}}{1 - \frac{3}{2} e^{i\pi/3}}\right| = \left|\frac{\left(2 + 3\cos\frac{\pi}{3}\right) + 3i\sin\frac{\pi}{3}}{2 - 3\cos\frac{\pi}{3} - 3i\sin\frac{\pi}{3}}\right|$$

$$= \sqrt{\frac{\left(\frac{7}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}} = \sqrt{\frac{49 + 27}{1 + 27}} = \frac{\sqrt{133}}{7}$$

14. $z_1 z_2 z_3 = -c$

$\Rightarrow \quad 1 = |c| \quad \Rightarrow |c| = 1$

$|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

$|a| \le 3$

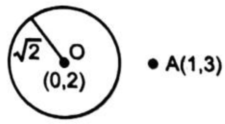
$|b| = |z_1 z_2 + z_2 z_3 + z_3 z_1| \le |z_1 z_2| + |z_2 z_3| + |z_3 z_1|$

$\Rightarrow \quad |b| \le 3$

15. $\frac{1}{2} \le |z| \le 4$

$$\left|z + \frac{1}{z}\right| = \sqrt{\left(\left(r + \frac{1}{r}\right)\cos\theta\right)^2 + \left(\left(r - \frac{1}{r}\right)\sin\theta\right)^2} = \sqrt{r^2 + \frac{1}{r^2} + 2(\cos\theta - \sin\theta)}$$

16. $|3 + i(z - 1)| = |z - 1 - 3i|$



Maximum distance of A from $(z) = OA + r$

$$= \sqrt{1 + 1} + \sqrt{2} = 2\sqrt{2}$$

17. $x^2 - (\sqrt{2}i)x - 1 = 0$

$$x = \frac{\sqrt{2}i \pm \sqrt{-2 + 4}}{2} = \frac{1}{\sqrt{2}}(\pm 1 + i)$$

$x = cis\,\frac{\pi}{4}, cis\,\frac{3\pi}{4}$

$x^{2187} = cis\,\frac{3\pi}{4},\ cis\,\frac{\pi}{4}$

$\frac{1}{x^{2187}} = cis\left(\frac{-3\pi}{4}\right),\ \ cis\left(-\frac{\pi}{4}\right) \Rightarrow x^{2187} - \frac{1}{x^{2187}} = 2i\sin\frac{3\pi}{4}, 2i\sin\frac{\pi}{4} = \sqrt{2}\,i$

18. $1 \cdot \frac{(1+z^9)}{1+z} = 0, z \neq -1$

$\Rightarrow \quad z^9 = -1$

$\Rightarrow \quad re^{i\theta} = e^{\frac{i(2n+1)\pi}{9}}, n = 1, 2, ...8$

19. Let $P(re^{i\alpha})$ & $Q(re^{i\beta})$

Point of intersection of tangents at 'α', 'β' to circle $x^2 + y^2 = r^2$ is

$$\left(r \cdot \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} - i\frac{r\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \right) = \frac{re^{i\left(\frac{\alpha+\beta}{2}\right)}}{\cos\frac{\alpha-\beta}{2}} = \frac{2\omega_1\omega_2}{\omega_1 + \omega_2}$$

20. $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2 = 2(4 + 9 + 16) - 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

where $\vec{a}, \vec{b}, \vec{c}$ are position vectors of points z_1, z_2, z_3

$\Rightarrow$ Maximum value $= 58 - 2(6 + 12 + 8)\left(-\frac{1}{2}\right) = 84$

21. We have $\quad Z = \frac{7+i}{3+4i}$

Simplifying (*i.e.*, rationalizing the denominator), we get

$$\frac{7+i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{21 + 4 - 28i + 3i}{9 + 16}$$

$$= \frac{25 - 25i}{25} = 1 - i$$

Therefore, $\quad \left(\frac{7+i}{3+4i}\right)^{14} = (1-i)^{14}$

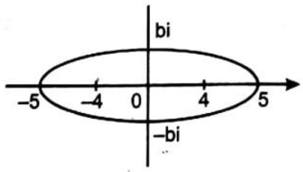
$$= [(1-i)^2]^7 = (1 + i^2 - 2i)^7$$

$$= (+2^7)i$$

22. $|Z - 4| + |Z + 4| = 10$

$PS + PS' = 2a$

which implies that foci at 4 and −4 and $a = 5$ as shown in the following figure.



Now, $b^2 = 25(1-e^2) = 25-(5e)^2$

$= 25-16 = 9$

$\Rightarrow \quad b = 3$

Z lies on the ellipse circumference $|Z|$ denotes the distance from the origin. Therefore,

$$|Z|_{max} = 5$$
$$|Z|_{min} = 3$$

Thus, the difference between the maximum and the minimum values of $|Z|$ is

$$|Z|_{max} - |Z|_{min} = 5-3 = 2$$

Exercise-2 : One or More than One Answer is/are Correct

1. Let $z_1 = re^{i\theta}$ and $z_2 = re^{i\phi}$

$|z_1 + z_2| = |z_1|$

$\Rightarrow \quad |e^{i\theta} + e^{i\phi}| = |e^{i\theta}| = 1$

$\Rightarrow \quad (\cos\theta + \cos\phi)^2 + (\sin\theta + \sin\phi)^2 = 1$

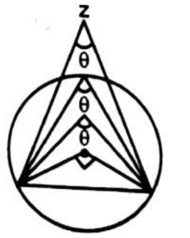
$\Rightarrow \quad \cos(\theta-\phi) = -\frac{1}{2} \quad \Rightarrow \quad \theta - \phi = \frac{2\pi}{3}$ or $-\frac{2\pi}{3}$

$\frac{z_1}{z_2} = e^{i(\theta-\phi)} = e^{i2\pi/3}$ or $e^{-i2\pi/3}$

2. (a) If $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ then z_1 and z_2 subtend right-angle at circumcentre origin.

$\therefore$ the chord joining z_1 and z_2 will subtend an angle θ at 'z' such that

$$\begin{cases} \theta = \pi/4 & \text{if} \quad |z| = 1 \\ \theta < \pi/4 & \text{if} \quad |z| > 1 \\ \theta > \pi/4 & \text{if} \quad |z| < 1 \end{cases}$$

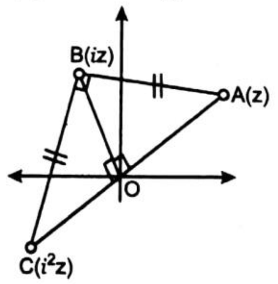


(b) $|z_1z_2 + z_2z_3 + z_3z_1| = |z_1|\cdot|z_2|\cdot|z_3|\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$

$= |\overline{z_1 + z_2 + z_3}|$

(c) $\left(\frac{(z_1+z_2)(z_2+z_3)(z_3+z_1)}{z_1z_2z_3}\right) = \left(\overline{\frac{(z_1+z_2)(z_2+z_3)(z_3+z_1)}{z_1z_2z_3}}\right)$

(d) The triangle formed by joining z_1, z_3 and z_2 is isosceles and right angled at z_3.

3. 

Method I : Multiplying a complex number by i rotates a vector for z in the anticlockwise direction by an angle of 90°.

$\therefore$ $$\angle AOB = \angle BOC = 90°$$

As shown in figure, the ΔABC is a right angled isosceles triangle.

Method II : Let z, iz, i^2z are vertices A, B and C of the triangle ABC.

$\therefore$ $$|AB| = |BC| \text{ also } |AB|^2 + |BC|^2 = |AC|^2$$

Since, $$|AB| = |BC| \text{ also } |AB|^2 + |BC|^2 = |AC|^2$$

$\therefore$ the ΔABC is a right angled isosceles triangle.

7. $(z+i)^4 = 1+i$

$$z = -i + 2^{1/8}\cos\left(\frac{\pi}{8} + \frac{2m\pi}{4}\right)$$

$$\text{Square side length} = \left(\frac{2^{1/8}\cdot 2}{\sqrt{2}}\right)$$

8. $z = 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$$\text{Roots} = 4^{1/4}\cos\left(\frac{2m\pi}{4} - \frac{60°}{4}\right)$$

$m = 0, 1, 2, 3 \qquad m = 1$

9. $a + b\omega + c\omega^2 = \alpha$

$a + b\omega^2 + c\omega = \overline{\alpha}$

$$|\alpha| = 1 \Rightarrow \left|a + b\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) + c\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\right| = 1$$

10. Check option for $z = \omega$

$\omega^{62} + \omega + 1 = 0 \qquad \omega^2 + \omega + 1 = 0$

$\omega^{155} + \omega + 1 = 0 \qquad \omega^2 + \omega + 1 = 0$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

Sol. $f(z)+\overline{f(z)}=f(\bar{z})+\overline{f(\bar{z})}$

$(\alpha z+\beta)+(\bar{\alpha}\,\bar{z}+\bar{\beta})=\alpha\,\bar{z}+\beta+\bar{\alpha}\,z+\bar{\beta}$

$\Rightarrow (\alpha-\bar{\alpha})(z-\bar{z})=0$

$\Rightarrow \text{Im}(\alpha)=0 \qquad (\text{Im}(z)\neq 0)$

$f(z)+\overline{f(z)}=0$

$\Rightarrow \alpha(z+\bar{z})+(\beta+\bar{\beta})=0 \qquad (\because \alpha=\bar{\alpha})$

$\Rightarrow \text{Re}(\beta)=0 \qquad (\text{Re}(z)=0)$

$|f(z)|^2>(z+1)^2$

$\Rightarrow \alpha^2z^2+\beta^2>z^2+2z+1$

$\Rightarrow (\alpha^2-1)z^2-2z+(\beta^2-1)>0\ \forall\ z\in R$

Paragraph for Question Nos. 3 to 5

Sol. $|\alpha-\beta|=2\sqrt{7}$

$\Rightarrow |(\alpha+\beta)^2-4\alpha\beta|=28$

$\Rightarrow |z_1^2-4(z_2+m)|=28$

$\Rightarrow |m-(4+5i)|=7$

greatest $(|m|)=\sqrt{16+25}+7$

least $|m|=7-\sqrt{16+25}$

Paragraph for Question Nos. 6 to 7

Sol. $C_1:|z-z_1|^2+|z-z_2|^2=10 \Rightarrow C_1:(x-5)^2+y^2=1$

$C_2:|z-z_1|^2+|z-z_2|^2=16 \Rightarrow C_2:(x-5)^2+y^2=4$

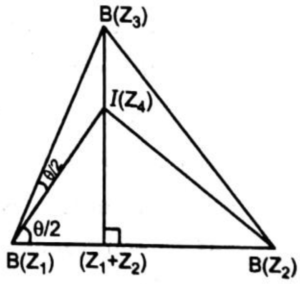
Paragraph for Question Nos. 8 to 9

Sol. $\dfrac{Z_2-Z_1}{|Z_2-Z_1|}=\dfrac{Z_4-Z_1}{|Z_4-Z_1|}\cdot e^{i\theta/2}, \quad \dfrac{Z_3-Z_1}{|Z_3-Z_1|}=\dfrac{Z_4-Z_1}{|Z_4-Z_1|}e^{i\theta/2}$

$\Rightarrow \dfrac{(Z_2-Z_1)(Z_3-Z_1)}{|Z_2-Z_1||Z_3-Z_1|}=\dfrac{(Z_4-Z_1)^2}{|Z_4-Z_1|^2}e^{\theta i}$

$\Rightarrow \dfrac{(Z_2-Z_1)(Z_3-Z_1)}{(Z_4-Z_1)^2}=\dfrac{AB\cdot AC}{(IA)^2}$

Similarly, others.



Exercise-4 : Matching Type Problems

1. Let $BC = n$, $CA = n+1$, $AB = n+2$

(A) $\left|\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\right| = \left|2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\right| = \angle C = 2\angle A$

$\therefore \quad \frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 2A}{n+2} = \frac{\sin A}{n}$

$\Rightarrow \quad \cos A = \frac{n+2}{2n} \Rightarrow \frac{(n+2)^2 + (n+1)^2 - n^2}{2(n+2)(n+1)} = \frac{n+2}{2n}$

$\Rightarrow \quad n(n^2 + n + 5) = (n^2 + 3n + 2)(n+2) \Rightarrow n^2 - 3n - 4 = 0 \Rightarrow n = 4$

$\therefore \quad$ biggest side $= n + 2 = 6$

(B) $(\vec{c} - \vec{a}) \cdot (\vec{b} - \vec{c}) = 0 \Rightarrow \angle C = 90° \Rightarrow a^2 + b^2 = c^2$

$\Rightarrow \quad n^2 + (n+1)^2 = (n+2)^2 \Rightarrow n = 3$

$\therefore \quad$ Area $= \frac{1}{2} \cdot 3 \cdot 4 = 6 = \Delta$

$\therefore \quad |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 2\Delta = 12$

(C) $\left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right| = \tan A = \frac{4}{3} \qquad \therefore \cos A = \frac{3}{5}$

$\therefore \quad \frac{(n+2)^2 + (n+1)^2 - n^2}{2(n+2)(n+1)} = \frac{3}{5}$

$\Rightarrow \quad 5(n^2 + 6n + 5) = 6(n^2 + 3n + 2)$

$\Rightarrow \quad n^2 - 12n - 13 = 0 \Rightarrow n = 13$

$\therefore \quad S - c = \frac{1}{2}(a + b - c) = \frac{1}{2}(13 + 14 - 15) = 6$

(D) Altitudes are in H.P. $\Leftrightarrow$ sides are in A.P.

Also, $b > a + c, a > b + c, c > a + b \Rightarrow$ least value of $a = 2$

$\therefore$ least value of $b = 3$

3. (A) $\{0, 1, \omega + 1\}^m = \{0, 1, -\omega^2\}^m$

$0, 1, -1, -\omega^2, -\omega, \omega$

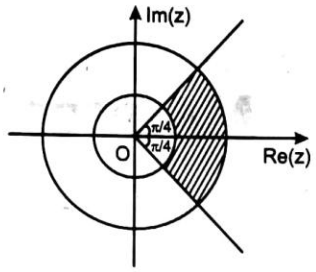
(B) $2\omega, (x^2 - x + 10) = 0 \qquad$ roots are $2 + 3\omega, \; 2 + 3\omega^2$

Last number is 3.

(C) Central angle $= 60° \qquad$ Equilateral Δ

(D) Put $z = 1 \qquad z_1 = 1, \; z_2 = \omega, \; z_3 = \omega^2$

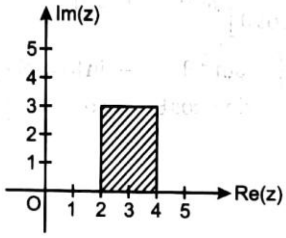
Exercise-5 : Subjective Type Problems

1. 

$2 \le |z| \le 4$

Probability $= \frac{1}{4}$

2.



3. $z + \bar{z} = 2|z-1| \quad \Rightarrow y^2 = 2x - 1$

$\arg(z_1 - z_2) = \frac{\pi}{4} \Rightarrow y_1 - y_2 = x_1 - x_2$

$y_1^2 - y_2^2 = 2(x_1 - x_2) = 2(y_1 - y_2) \quad \Rightarrow y_1 + y_2 = 2 \quad (y_1 \ne y_2)$

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Chapter 12 – Matrices

12 MATRICES

Exercise-1 : Single Choice Problems

1. $A = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} [\cos\theta \quad \sin\theta] + \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} [\sin\theta \quad -\cos\theta]$

$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} = I$

2. $|A| = \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 1$

$A^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

det (adj (adj (A))) $= |A|^4 = 27^4$

$$\left\{ \frac{27^4}{5} \right\} = \frac{1}{5}$$

4. $A^{-1}B^{-1} = B^{-1}A^{-1} \Rightarrow C = (A^{-1} + B^{-1})^5 = (I)^5$

5. $A^4 = I \Rightarrow A(A^3) = I$

7. (adj A) $A = |A| I$

$|A| = xyz - 8x - 4y - 3z + 28 = 2\lambda - \lambda = \lambda$

8. $(x-2) + (x^2 - x + 3) + (x - 7) = 0$

$x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$

9. $A=\begin{bmatrix}-1 & 3\\ 3 & 0\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{9}\begin{bmatrix}0 & 3\\ 3 & 1\end{bmatrix}$

10. $\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\begin{bmatrix}\cos^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{bmatrix}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

$\Rightarrow \quad a=\cos 2\theta,\ b=\sin 2\theta$

11. $P^2=I-P$

or $P^3=P-P^2=2P-I$

or $P^4=2I-3P$

or $P^5=-3I+5P$

or $P^6=5I-8P$

12. $|\text{adj}(\text{adj}(A))|=|A|^{(n-1)^2}$

$\Rightarrow \quad |A|=x+y+z=12$

$x\geq 1, y\geq 1, z\geq 1$

$\Rightarrow \quad {}^{11}C_2=55$

13. Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$; $\text{adj}(A)=\begin{bmatrix}d & -c\\ -b & a\end{bmatrix}^T=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$; $\text{adj}(\text{adj}(A))=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

14. $M=A^{2m}\cdot A^{-1}$

$M=\dfrac{A^{2m+1}}{a^2+b^2}$

If $A^2=(a^2+b^2)\cdot I \Rightarrow A^{2m}=(a^2+b^2)^m\cdot I$

$A^{2m+1}=(a^2+b^2)^m\cdot A$

15. $A^2+5A+6I=I$

$(A+2I)(A+3I)=I$

$\Rightarrow$ $A+2I$ and $A+3I$ are inverse of each other.

16. $AB=\begin{bmatrix}3 & -5\\ 7 & -12\end{bmatrix}\begin{bmatrix}12 & -5\\ 7 & -3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

17. $\text{adj}(A)=\begin{bmatrix}3 & -2\\ -2 & 2\end{bmatrix}$

18. $AA^1=I$

$$\begin{bmatrix}\cos\theta & 2\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ 2\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos^2\theta+4\sin^2\theta & 3\sin\theta\cos\theta\\ 3\sin\theta\cos\theta & \sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow \sin\theta=0$$

20. If $A=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$, $P=\begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}$, $Q=P^TAP$, we have

$$PQ^{2014}P^T=\underbrace{P(P^TAP)(P^TAP)\ldots(P^TAP)P^T}_{2014 \text{ times}}$$

$$=(PP^T)A(PP^T)A(PP^T)\ldots(PP^T)A(PP^T)$$

Matrix multiplication is associative.

$$PP^T=\begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=I_2$$

Hence, $PQ^{2014}P^T=A^{2014}$

$$A=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}$$

$$A^3=\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 6\\0 & 1\end{pmatrix}$$

$$A^4=\begin{pmatrix}1 & 6\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 8\\0 & 1\end{pmatrix}$$

$$\Rightarrow \quad A^n=\begin{pmatrix}1 & 2n\\0 & 1\end{pmatrix} \text{ and } A^{2014}=\begin{pmatrix}1 & 4028\\0 & 1\end{pmatrix}$$

21. $\left|adj\left(\frac{M}{2}\right)\right|=\left|\frac{M}{2}\right|^2=\left(\frac{1}{8}|M|\right)^2$

22. $|A^{-1}|=\frac{1}{|A|}=\frac{1}{5}$

$|(AB)^T|=|AB|=|A\cdot(adj\ A)|=|A|\cdot|adj\,(A)|=5\times5^2=5^3$

$\therefore ||A^{-1}|(AB)^T|=|\frac{1}{5}(AB)^T|=\frac{1}{5^3}|AB|=1$

Exercise-2 : One or More than One Answer is/are Correct

3. $A_\alpha A_\beta=A_{\alpha+\beta}$

Also, $A_0=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=I$

and $A_\alpha A_{-\alpha}=A_{\alpha-\alpha}=A_0=I$

we get $A_\alpha^{-1} = A_{-\alpha}$

However, $A_\alpha^{-1} = -A_\alpha$ and $A_\alpha^2 = -I$ do not hold.

4. $A(A^2 - I) - 2(A^2 - I) = 0$

$(A^2 - I)(A - 2I) = 0$

Exercise-3 : Matching Type Problems

1. (A) Possible non-negative value of $|A| = 2, 4, 8$

(B) Sum is 0.

(C) $|\text{adj}(\text{adj}(\text{adj } A))| = |A|$

least absolute value of $|A| = 2$

$\Rightarrow \quad |A| = \pm 2$

(D) least $|A| = -8$

$$|4A^{-1}| = \frac{16}{|A|} = -2$$

2. (A) Since A is idempotent, $A^2 = A^3 = A^4 = \ldots\ldots = A$. Now,

$$\begin{aligned}(A + I)^n &= I + {}^nC_1A + {}^nC_2A^2 + \ldots\ldots + {}^nC_nA^n \\ &= I + {}^nC_1A + {}^nC_2A + \ldots\ldots + {}^nC_nA \\ &= I + ({}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n)A \\ &= I + (2^n - 1)A\end{aligned}$$

$\Rightarrow \quad 2^n - 1 = 127 \Rightarrow n = 7$

(B) We have,

$$\begin{aligned}&(I - A)(I + A + A^2 + \ldots\ldots + A^7) \\ &= I + A + A^2 + \ldots\ldots + A^7 + (-A - A^2 - A^3 - A^4 \ldots\ldots - A^8) \\ &= I - A^8 \\ &= I \qquad \text{(if } A^8 = 0)\end{aligned}$$

(C) Here matrix A is skew-symmetric and since $|A| = |A^T| = (-1)^n|A|$, so $|A|(1 - (-1)^n) = 0$. As n is odd, hence $|A| = 0$. Hence A is singular.

(D) If A is symmetric, A^{-1} is also symmetric for matrix of any order.

5. (A) $\displaystyle \frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{\frac{r}{n}}}\right) = \int_0^1 \frac{1}{\sqrt{x}}\,dx$

(B) $D = 4\cos t \cos 2t$

(C) $3x^2 + 2px + g < 0$

$f\left(-\frac{5}{3}\right) = 0 \quad f(-1) = 0$

(D) $(2^x - 2)^2 + 1 + ||b-1|-3| = |\sin y|$

$b - 1 = \pm 3$

$|\sin y| = 1$

Exercise-4 : Subjective Type Problems

1. $(AB)^2 = AB \cdot AB = A^3B^2$

$(AB)^3 = (AB)^2 \cdot AB = A^3B^2 \cdot AB = A^7B^3$

$(AB)^4 = (AB)^3 \cdot AB = A^7B^3 \cdot AB = A^{15}B^4 \Rightarrow (AB)^{10} = A^{1023}B^{10}$

2. $l = \lim_{n\to\infty} 18\left(\frac{3}{3^2} + \frac{3^2}{3^4} + \frac{3^3}{3^6} + \ldots\ldots\right) = 18 \times \frac{1}{3\left(1 - \frac{1}{3}\right)} = 9$

$m = \lim_{n\to\infty} 12\left(\frac{2}{2^2} + \frac{2^2}{2^4} + \ldots\ldots\right) = 12 \times \frac{1}{2\left(1 - \frac{1}{2}\right)} = 12$

4. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1c_2b_3 \quad$ ...six elements

All cannot be simultaneously 1.

5. First element of matrix $A_{10} = 286$ (10^{th} of sequence 1, 2, 6, 15, ...)

Trace of $A_{10} = 286 + 297 + 308 + 319 + \ldots + 385 = 3055$

□□□

Chapter 13 – Permutation and Combinations

13 PERMUTATION AND COMBINATIONS

Exercise-1 : Single Choice Problems

1. 7 _ _ = 81 (9, 9); _ 7 _ = 72 (8, 9); _ _ 7 = 72 (8, 9)

2. $\left(\frac{8!}{3!3!2!2!}\right)\times {}^2C_1 \times 3! + \left(\frac{8!}{3!2!2!2!}\right)\times 3! = 8400$

3. Number of ways $= 6 \times \left(\frac{3!}{2!}\times 3!\right) = 108$

4. ${}^4C_1 \times \frac{5!}{2!} = 240$

5. ${}^6C_2 \times 1 \times 4! = 360$

6. $x^2 - 5x + 3 = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$\Rightarrow \quad \alpha + \beta = 5, \quad \alpha\beta = 3$

Sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{19}{3}$

7. ${}^5C_4 \times {}^8C_6 + {}^5C_5 \times {}^8C_5 = 196$

8. $(1 + 2 + 3 + \ldots\ldots + 22)\ {}^{21}C_{10}$

9. $x = \frac{2009 \times 2008 \times 2007 + 1}{2008 \times 2007 \times 2007} = 2008 + \frac{1}{2009 \times 2007}$

$\Rightarrow \quad [x] = 2008$

10. $N = p_1^n p_2 p_3 \ldots\ldots p_{m+1}$

No. of factors $= (n+1)\,2^m$

11. Number of ways $= (11)! \times 2^{12}$

12. $\underset{5}{\uparrow}\ \underset{5}{\uparrow}$ — —

$m = 5 \times 5 \times {}^8C_2 \times 2!$

$\underset{4}{\uparrow}\ \underset{5}{\uparrow}$ — —

$n = 4 \times 5 \times {}^8C_2 \times 2!$

13. Three different digits (not including zero)

$${}^9C_3 \times 2!$$

Two digits (not including zero)

$${}^9C_2 \times 2$$

Three digits (including zero)

$${}^9C_2 \times 1$$

14. Let no. of elements in $A = n$

No. of elements in $B = m$

$$2^n - 2^m = 1920 = 2^7 \times 15$$

$$\Rightarrow \quad n = 11,\ m = 7$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 15$$

15. $C\ldots\ldots\ldots = 4! = 24$

$D\ldots\ldots\ldots = 4! = 24$

$M\ldots\ldots\ldots = 4! = 24$

Ⓢ$C\ldots\ldots = 3! = 6$

Ⓢ$D\ldots\ldots = 3! = 6$

Ⓢ Ⓜ Ⓒ $D\ W = 1$

Ⓢ Ⓜ Ⓒ $W\ D = 1$

16. $P = $ All A's together $= \dfrac{5!}{3!}$; $\qquad Q = $ All B's together $= \dfrac{6!}{4!}$

$$n(P \cap Q) = 3!; \qquad n(P \cup Q) = \frac{5!}{3!} + \frac{6!}{4!} - 3! = 50 - 6 = 44$$

17. $5^6 \times 6^7 \times 7^8 \times 8^9 \times 9^{10} \times 10^{11} \times \ldots\ldots \times 30^{31}$

No. of zero's = no. of 5's

$$= 6 + 11 + 16 + 21 + (2 \times 26) + 31 = 137$$

18. $(x - y)(x + y) = 10 \times 337$

$\Rightarrow \quad x - y = 10$ and $x + y = 337$

$$x = \frac{347}{2} \qquad \text{(not possible)}$$

19. Total number of different things $= n + 2$

20. Let the numbers are $10-d, 10, 10+d$.

$$d \in \{-9, -8, -7, \ldots\ldots, 7, 8, 9\}$$

22. $m = 2 \times 5! \times 5!$

$n = 4! \times 5!$

23. Total ways $= 4 \times 4! = 96$

25. ${}^4C_2 \times 5^2 \times (21)^2 = 66150$

26. Total all letters are different.

$\Rightarrow \quad 10^5 - {}^{10}C_5 \times 5! = 69760$

29. $M = 1440$

$M = 2^5 \cdot 3^2 \cdot 5$

No. of divisions $= 6 \times 3 \times 2 = 36$

$P =$ Product of divisors $= (1440)^{18}$

$P = 2^{90} \cdot 3^{36} \cdot 5^{18}$

Hence, $x = 30$

30. **Case-1 :** All digits same $= 9$

Case-2 : Excluding zero :

(i) No's having 3 digits same : ${}^9C_2 \times {}^2C_1 \times \dfrac{4!}{3!} = 288$

(ii) No's having 2 digits same, 2 other same : ${}^9C_2 \times \dfrac{4!}{2!2!} = 216$

Case-3 : Including zero :

(i) No's having 3 zero's : 9

(ii) No's having 2 zero's : ${}^9C_1 \times \dfrac{3!}{2!} = 27$

(iii) No's having 1 zero $= {}^9C_1 \times \dfrac{3!}{2!} = 27$

Hence, total no's $= 576$

31. **Case-I :** When two T's contain exactly one vowel between them,

$$5! \times ({}^5C_1 \times {}^5C_4 \times 4!) = 15 \times 5! \times 5!$$

Case-II : When two T's also contain consonant between them,

$$4! \times ({}^5C_2) \times ({}^7C_5 \times 5!) = 42 \times 5! \times 5!$$

32. $666660 \rightarrow 6$

$666633 \rightarrow \dfrac{6!}{4!2!}$

$666642 \rightarrow \dfrac{6!}{4!}$

$666444 \rightarrow \dfrac{6!}{3!3!}$

33. Five 4 runs + one 0 run $= \dfrac{6!}{5!}$

Four 4 runs + two 2 runs $= \dfrac{6!}{4!2!}$

Three 4 runs + two 3 runs + one 2 runs $= \dfrac{6!}{3!2!}$

Two 4 runs + four 3 runs $= \dfrac{6!}{2!4!}$

$\Rightarrow \quad N = 96$

34. ${}^{7}C_2 = 21$

35. $x_1 + x_2 + x_3 + x_4 + x_5 = 101$

Let $x_1 = 2k_1 + 1, x_2 = 2k_2 + 1, x_3 = 2k_3 + 1, x_4 = 2k_4 + 1, x_5 = 2k_5 + 1$

$\Rightarrow \quad k_1 + k_2 + k_3 + k_4 + k_5 = 48; \quad {}^{48+5-1}C_{5-1}$

36. Total ways = (largest number is 4)

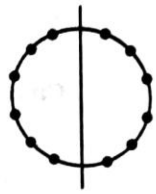
$6^4 - (4^4 - 3^4) = 1121$

37. ${}^{6}C_3 \times 4!$

38. If two points are selected from one side of main diagonal $= {}^{6}C_2$.

Then other two points are selected on other side of main diagonal = 1.

Total ways $= {}^{6}C_2 \times 1 = 15$



39. $(9 - x_1) + (9 - x_2) + (9 - x_3) + (9 - x_4) + (9 - x_5) = 43$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 + x_5 = 2$

Number of ways $= {}^{2+5-1}C_{5-1} = {}^{6}C_4 = 15$

Exercise-2 : One or More than One Answer Is/are Correct

1. Case-I : All five letters are different.

$= 5!$

Case-II : Two letters are same and remaining are different.

$${}^{3}C_1 \times {}^{4}C_3 \times \frac{5!}{2!} = 720$$

Case-III : Two alike, two other alike and remaining different.

$${}^{3}C_2 \times {}^{3}C_1 \times \frac{5!}{2!2!} = 270$$

Total number of words = 1110

2. $\sum_{k=0}^{100} {}^{100}C_k (x-2)^{100-k} \cdot 3^k = (x+1)^{100}$

Coeff. of $x^{50} = {}^{100}C_{50}$

3. $\dfrac{\text{Total} - \text{Row 1} - \text{Row 2}}{\lfloor 2}$ $\{\lfloor 2 \text{ for } N\}$

$\dfrac{{}^8C_5 \lfloor 6 - \lfloor 6 - \lfloor 6}{\lfloor 2}$

4. = (four odd) + (4 even) + (3 even + 1 odd) + (2 even + 2 odd)

$= {}^5C_4 \times 4! + {}^4C_4 \times 4! + {}^4C_3 \times {}^5C_1 \times 4! + {}^4C_2 \times {}^5C_2 \times 4 \times 4$

$= 1584$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. 0

2. Digit 6 always come at last three place digit 5 always come at last four place and digit 4 always come at last five place.

$${}^3C_1 \times {}^3C_1 \times {}^3C_1 \times 3! = 162$$

Exercise-4 : Matching Type Problems

1. (A) $\dfrac{6!}{2!} \times {}^7C_2 = 7560$ (B) $5! \times {}^6C_2 = 1800$

(C) $7560 - 1800 = 5760$ (D) $4! \times {}^5C_4 \cdot \dfrac{4!}{2!2!} = 720$

2. (A) Total ways – (No repeating letter is at odd position)

$$\frac{11!}{2!2!2!} - 0 = \frac{11!}{(2!)^3}$$

(B) $\dfrac{7!}{2!2!} \times {}^8C_4 \times \dfrac{4!}{2!} = 210 \times 7!$

(C) (MM) (TT) $HEICS$

$7! \times {}^8C_2 \times 1 = 28 \times 7!$

(D) $\left(\dfrac{4!}{2!}\right) \times \left(\dfrac{7!}{2!2!}\right) = \dfrac{4!\,7!}{(2!)^3}$

Exercise-5 : Subjective Type Problems

1. ${}^9C_4 \times {}^5C_4 = 630$

2. ${}^9C_2 \times \dfrac{7!}{2!2!} = \dfrac{9!}{8}$

4. ${}^{10}C_3 - {}^8C_3 = 64$

5. **Case-I:** If Ravi is include.

$${}^7C_5 \times {}^9C_8 = 189$$

Case-II: If Ravi is not include.

$${}^7C_6 \times [{}^8C_7 + {}^9C_8] = 119$$

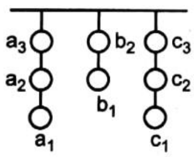
Total number of ways $= 308$

6. ${}^6C_4 - {}^4C_2 = 9$

7. $5! - (1 + {}^5C_2 \times 1) = 109$

8. Let other two sides are a and b.

$\therefore \quad a + b > 11 \qquad 0 < a \le 11, \quad 0 < b \le 11$

9. 

$(a_1, a_2, a_3), (b_1, b_2)$ and (c_1, c_2, c_3) are alike things so these can be arranged is

$$\frac{8!}{2!3!3!} = \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{2 \times 6} = 560$$

10. ${}^nC_2 - n = 14 \Rightarrow n = 7$

11. $x_1 + x_2 + x_3 + \ldots\ldots + x_7 + x_8 = 93$

$x_1 \ge 0,\ x_2 \ge 6,\ x_3 \ge 6, \ldots\ldots x_7 \ge 6,\ x_8 \ge 0$

$x_1 + x_2' + x_3' + \ldots\ldots + x_7' + x_8 = 57$

No. of ways $= {}^{64}C_7$

12. ${}^4C_4({}^2C_1)^4 = 16$

13. Let x_1 objects of one type

x_2 objects of second type

x_3 objects of third type

$x_1 + x_2 + x_3 = 3n$

$0 \le x_1 \le 2n, 0 \le x_2 \le 2n, 0 \le x_3 \le 2n$

Number of ways = ${}^{3n+2}C_2 - 3 \times {}^{n+1}C_2 = 3n^2 + 3n + 1$

14. $x + y + z + w = 15$

$x \ge 0, y \ge 6, z \ge 2, w \ge 1$

$x + y' + z' + w' = 6$

Number of ways = ${}^9C_3 = 84$

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Chapter 14 – Binomial Theorem

14 BINOMIAL THEOREM

Exercise-1 : Single Choice Problems

1. Let $x = 2^{\frac{153}{2}}$

$\alpha = x^2 + \sqrt{2}\,x + 1$

Let $y = 2^{204}$

$\beta = y^2 - y + 1$

$N = x^{16} - 1$

$N = (x^4 - 1)(x^4 + 1)(x^8 + 1)$

$N = (x^4 - 1)(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)(x^8 + 1)$

$N = y^6 - 1 = (y^3 - 1)(y^3 + 1)$

$= (y^3 - 1)(y + 1)(y^2 - y + 1)$

3. ${}^4C_2\alpha^2 = -\,{}^6C_3\alpha^3$

$\Rightarrow \quad \alpha = -\dfrac{3}{10}$

5. $\alpha_n = (2 + \sqrt{3})^n$

Let $\alpha'_n = (2 - \sqrt{3})^n \Rightarrow \alpha_n + \alpha'_n = \text{integer}$

$\Rightarrow \quad [\alpha_n] + \{\alpha_n\} + \alpha'_n = \text{integer} \Rightarrow \{\alpha_n\} = 1 - \alpha'_n$

So, $\lim\limits_{n\to\infty}(\alpha_n - [\alpha_n]) = \lim\limits_{n\to\infty}[1 - (2 - \sqrt{3})^n] = 1 - 0 = 0 \qquad (\because 0 > \{\alpha_n\}, \alpha'_n < 1)$

6. $N = {}^{20}C_7 - {}^{20}C_8 + {}^{20}C_9 - {}^{20}C_{10} + \ldots\ldots - {}^{20}C_{20}$

$= ({}^{20}C_7 + {}^{20}C_9 + {}^{20}C_{11} + \ldots\ldots + {}^{20}C_{19}) - ({}^{20}C_8 + {}^{20}C_{10} + \ldots\ldots + {}^{20}C_{20})$

$= ({}^{20}C_0 + {}^{20}C_2 + {}^{20}C_4 + {}^{20}C_6) - ({}^{20}C_1 + {}^{20}C_3 + {}^{20}C_5)$

$= (1 + 190 + 4845 + 38760) - (20 + 1140 + 15504)$

$= 43796 - 16664 = 27132 = 3 \times 4 \times 7 \times 19 \times 17$

7. $\log_2\left[1 + \dfrac{1}{2}(2^{12} - 2)\right] = \log_2 2^{11} = 11 \qquad [\because \Sigma\,{}^nC_r = 2^n]$

8. $T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r = {}^{12}C_r \cdot x^{12-r} \cdot \left(\dfrac{1}{x^3}\right)^r$

$$12-4r=0 \Rightarrow r=3$$

$$T_4={}^{12}C_3=\frac{12\times11\times10}{3\cdot2\cdot1}=220$$

9. $$\frac{3}{4!}+\frac{4}{5!}+\frac{5}{6!}+\ldots\ldots=\frac{1}{3!}-\frac{1}{(k+3)!}$$

$$\sum_{r=3}^{52}\frac{r}{(r+1)!}=\sum_{r=3}^{52}\left[\frac{1}{r!}-\frac{1}{(r+1)!}\right]=\frac{1}{3!}-\frac{1}{53!}$$

$\Rightarrow$ $$k=50$$

10. $$f(x)=\sum_{r=1}^{n}[(r+1)^2\ {}^nC_r-r^2\ {}^nC_{r-1}]$$

$$f(n)=(n+1)^2-1$$

$$f(30)=960$$

12. ${}^nC_1\cdot\alpha+{}^nC_2\cdot\alpha^2+{}^nC_3\cdot\alpha^3+\ldots\ldots{}^nC_n\cdot\alpha^n=(1+\alpha)^n-1$

$\left(\text{where } \alpha=e^{\frac{2\pi i}{n}}=\frac{\alpha_2}{\alpha_1}\right)$

13. $2^{30}\cdot3^{20}=2^{10}\cdot(6)^{20}=1024(7-1)^{20}=1024(7K+1)=7k'+1024=7k'+1022+2$

14. ${}^{26}C_0+{}^{26}C_1+{}^{26}C_2+\ldots\ldots+{}^{26}C_{26}=2^{26}$

$\Rightarrow\quad 2({}^{26}C_0+{}^{26}C_1+\ldots\ldots+{}^{26}C_{13})=2^{26}+{}^{26}C_{13}$

15. $(1+x+x^2)^n=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_{2n}x^{2n}$

differentiate w.r.t. x

$n(1+x+x^2)^{n-1}(1+2x)=a_1+2a_2x+3a_3x^2+\ldots+2n\cdot a_{2n}x^{2n-1}$

Put $x=1$ $\quad n\cdot3^n=a_1+2a_2+3a_3+\ldots+2n\,a_{2n}$ ...(1)

Put $x=\omega$ $\quad 0=a_1+2a_2\omega+3a_3\omega^2+\ldots+2n\,a_{2n}\omega^{2n-1}$ ...(2)

Put $x=\omega^2$ $\quad 0=a_1+2a_2\omega^2+3a_3\omega^4+\ldots+2n\,a_{2n}\omega^{4n-2}$ ...(3)

(1) + (2) + (3)

$$n\cdot3^{n-1}=a_1+4a_4+7a_7+10a_{10}+\ldots$$

16. ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$

${}^3C_0+{}^3C_1+{}^4C_2+{}^5C_3+\ldots+{}^{99}C_{97}={}^{100}C_{97}$

17. Last digit of $9!=0$

Last digit of $3^{9966}=9$

Hence last digit 9.

18. $x=T_7={}^nC_6(3^{1/3})^{n-6}\cdot(4^{-1/3})^6$

$y = T_{n-5} = {}^nC_{n-6}(3^{1/3})^6 \cdot (4^{-1/3})^{n-6}$

$y = 12x$

${}^nC_{n-6}(3^{1/3})^6(4^{-1/3})^{n-6} = 12 \cdot {}^nC_6(3^{1/3})^{n-6}(4^{-1/3})^6$

$\Rightarrow \quad 12 = (12^{1/3})^{12-n} \Rightarrow n = 9$

20. $t_{r+1} = {}^{15}C_r(x^2)^{15-r} \cdot \left(\frac{2}{x}\right)^r$

Coeff. of $x^{15} = {}^{15}C_5 \cdot 2^5$

Coeff. of $x^0 = {}^{15}C_{10} \cdot 2^{10}$

21. $(1+x)^2(1+y)^3(1+z)^4(1+w)^5$

General term $= {}^2C_a \; {}^3C_b \; {}^4C_d \; {}^5C_e x^{a+b+d+e}$

$$\sum_{a+b+d+e=12} {}^2C_a \times {}^3C_b \times {}^4C_d \times {}^5C_e = {}^{14}C_{12} \text{ or } {}^{14}C_{12} = \frac{14 \times 13}{2} = 91$$

22. $\sum_{r=0}^{n} r \cdot {}^nC_r + 2\sum_{r=0}^{n} \frac{1}{r+1} \cdot {}^nC_r; \quad n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \frac{2}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1}$

$\Rightarrow \quad n \cdot 2^{n-1} + \frac{2}{n+1} \cdot (2^{n+1} - 1)$

Exercise-2 : One or More than One Answer is/are Correct

1. $N = {}^{20}C_7 - {}^{20}C_8 + {}^{20}C_9 - {}^{20}C_{10} + \ldots\ldots - {}^{20}C_{20}$

$= ({}^{20}C_7 + {}^{20}C_9 + {}^{20}C_{11} + \ldots\ldots + {}^{20}C_{19}) - ({}^{20}C_8 + {}^{20}C_{10} + \ldots\ldots + {}^{20}C_{20})$

$= ({}^{20}C_{20} + {}^{20}C_2 + {}^{20}C_4 + {}^{20}C_6) - ({}^{20}C_1 + {}^{20}C_3 + {}^{20}C_5)$

2. For B and D put $x = 1, -1$

For A differentiate with respect to x then put $x = 0$

For C replace x with $\frac{1}{x}$

3. $\sum_{r=0}^{4}(-1)^r \, {}^{16}C_r = {}^{16}C_0 - {}^{16}C_1 + {}^{16}C_2 - {}^{16}C_3 + {}^{16}C_4 = 1365$

4. $2 \times \frac{1}{2} \times {}^nC_1 = 1 + \frac{1}{2^2} \times {}^nC_2 \Rightarrow n = 8, 1$

$$T_{r+1} = {}^8C_r\left(\frac{1}{2}\right)^r x^{\frac{16-3r}{4}} \Rightarrow r = 0, 4, 8$$

5. LHS $= (1 + 2x^2 + x^4)(1 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots)$

RHS $= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$

Comparing the coefficients of $x, x^2, x^3, \ldots\ldots$

Now, $2a_2 = a_1 + a_3$

$2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1)$

$2\dfrac{n(n-1)}{2} + 4 = 3n + \dfrac{n(n-1)(n-2)}{6}$

or $n^3 - 9n^2 + 26n - 24 = 0$

$\therefore$ $(n-2)(n^2 - 7n + 12) = 0$ $(\because 8 + 52 = 36 + 24)$

or $(n-2)(n-3)(n-4) = 0$

$\therefore$ $n = 2, 3, 4$

6. $\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} {}^nC_i \cdot {}^nC_j \cdot {}^nC_k = \left(\sum_{i=0}^{n} {}^nC_i\right)\left(\sum_{j=0}^{n} {}^nC_j\right)\left(\sum_{k=0}^{n} {}^nC_k\right) = 2^{3n}$

7. $({}^{100}C_6 + {}^{100}C_7) + 3({}^{100}C_7 + {}^{100}C_8) + 3({}^{100}C_8 + {}^{100}C_9) + ({}^{100}C_9 + {}^{100}C_{10})$

$= {}^{101}C_7 + 3\,{}^{101}C_8 + 3\,{}^{101}C_9 + {}^{101}C_{10}$

$({}^{101}C_7 + {}^{101}C_8) + 2({}^{101}C_8 + {}^{101}C_9) + ({}^{101}C_9 + {}^{101}C_{10}) = {}^{102}C_8 + 2\cdot{}^{102}C_9 + {}^{102}C_{10}$

$= ({}^{102}C_8 + {}^{102}C_9) + ({}^{102}C_9 + {}^{102}C_{10}) = {}^{103}C_9 + {}^{103}C_{10} = {}^{104}C_{10}$

8. $\dfrac{{}^{15}C_{2r}}{{}^{15}C_{2r+1}} > \dfrac{1}{2} \Rightarrow \dfrac{2r+1}{15-2r} > \dfrac{1}{2} \Rightarrow \dfrac{6r-13}{2r-15} < 0 \Rightarrow \dfrac{13}{6} < r < \dfrac{15}{2}$

9. $f(x) = 1 + x^{111} + x^{222} + \ldots + x^{999}$

if $f(x)$ is divided by $x+1$, then remainder $f(-1) = 0$

if $f(x)$ is divided by $x-1$, then remainder $f(1) = 10$

$f(x) = (1 + x^{222} + x^{444} + x^{666} + x^{888}) + x^{111}(1 + x^{222} + x^{444} + x^{666} + x^{888})$

$= (1 + x^{111})(1 + x^{222} + x^{444} + x^{666} + x^{888})$

Exercise-3 : Matching Type Problems

2. (B) $P = \sum_{r=0}^{n} {}^nC_r = 2^n$

$Q = \sum_{r=0}^{m} {}^mC_r (15)^r = (1+15)^m = 16^m$

$\Rightarrow$ $n = 4m$

(C) $1 + 6 + 120 + 56K$

Reminder $= 15$

3. (A) $\frac{a^2+b^2+ab}{a+b}=\frac{(a-b)(a^2+b^2+ab)}{(a-b)(a+b)}=\frac{a^3-b^3}{a^2-b^2}$

$\frac{4+\sqrt{3}}{\sqrt{3}+1}+\frac{8+\sqrt{15}}{\sqrt{5}+\sqrt{3}}+\frac{12+\sqrt{35}}{\sqrt{7}+\sqrt{5}}+\ldots=\frac{1}{2}((\sqrt{169})^3-1^3)=1098$

(B) $\frac{8}{5}(2\cos^2\theta-3\sin\theta)=\frac{8}{5}(-2\sin^2\theta-3\sin\theta+2)$

Greatest value $=5$ at $\sin\theta=-\frac{3}{4}$ $\quad (\because 4\le\theta\le 6)$

(C) Let $(\sqrt{3}+1)^6=I+f$

and $(\sqrt{3}-1)^6=f' \Rightarrow (\sqrt{3}+1)^6+(\sqrt{3}-1)^6=416=I+1$

$\Rightarrow \quad I=415=1\times 5\times 83$

(D) $(1+x)(1+x^2)\ldots(1+x^{128})=\frac{1-x^{256}}{1-x}=\frac{1-x^{n+1}}{1-x}$

$\Rightarrow \quad n+1=256$

Exercise-4 : Subjective Type Problems

2. Coefficient of $x^{60}=-6+5+8-6=1$

7. $(1+x)^{3n}={}^{3n}C_0+{}^{3n}C_1x+{}^{3n}C_2x^2+\ldots\ldots+{}^{3n}C_{3n}x^{3n}$

Put $x=1$ $\quad 2^{3n}={}^{3n}C_0+{}^{3n}C_1+{}^{3n}C_2+\ldots\ldots+{}^{3n}C_{3n}$

Put $x=\omega$ $\quad (-\omega^2)^{3n}={}^{3n}C_0+{}^{3n}C_1\omega+{}^{3n}C_2\omega^2+\ldots\ldots+{}^{3n}C_{3n}$

Put $x=\omega^2$ $\quad (-\omega)^{3n}={}^{3n}C_0+{}^{3n}C_1\omega^2+{}^{3n}C_2\omega^4+\ldots\ldots+{}^{3n}C_{3n}$

$2^{3n}+(-\omega^2)^{3n}+(-\omega)^{3n}=3[{}^{3n}C_0+{}^{3n}C_3+\ldots\ldots+{}^{3n}C_{3n}]$

10. $\sum_{K=1}^{5}{}^{20}C_{2K-1}=2^{18} \Rightarrow 2^{108}=2^3(2^5)^{21}=8(33-1)^{21}$

Remainder $=-8$ or 3

11. $f(n)={}^nC_0a^{n-1}-{}^nC_1a^{n-2}+\ldots\ldots$

$\Rightarrow \quad f(n)=\frac{(a-1)^n}{a}$

$f(2007)+f(2008)=3^7K$

$\Rightarrow \quad \frac{3^9+(a-1)3^9}{a}=3^7K \Rightarrow K=9$

13. $(360+1)^{44}-1={}^{44}C_0\cdot(360)^{44}+{}^{44}C_1\cdot(360)^{43}+\ldots+{}^{44}C_{43}\cdot(360)^1$

$=360[{}^{44}C_0\cdot(360)^{43}+{}^{44}C_1\cdot(360)^{42}+\ldots+{}^{44}C_{43}]$

14. $(3^{|x-2|} + (3^{|x-2|-9})^{1/5})^7$

$T_6 = {}^7C_5 \cdot (3^{|x-2|})^2 \cdot 3^{|x-2|-9} = 567$

$\Rightarrow \quad 3^{3|x-2|-9} = 27 \Rightarrow |x-2| = 4 \Rightarrow x = 6, -2$

15. $1 + \sum_{r=1}^{10} 3^r \cdot {}^{10}C_r + \sum_{r=1}^{10} r \cdot {}^{10}C_r$

$1 + ((1+3)^{10} - {}^{10}C_0) + 10 \cdot 2^9 = 4^{10} + 5 \cdot 2^{10} = 2^{10}(4^5 + 5)$

$\alpha = 1, \beta = 5$

if α, β lies between the roots of $f(x) = 0$

$f(1) < 0 \cap f(5) < 0$

$-k^2 < 0 \cap 16 - k^2 < 0$

16. $S_n = {}^nC_0\ {}^nC_1 + {}^nC_1\ {}^nC_2 + \ldots + {}^nC_{n-1} \cdot {}^nC_n = {}^{2n}C_{n-1}$

$S_{n+1} = {}^{2n+2}C_n$

$$\frac{S_{n+1}}{S_n} = \frac{{}^{2n+2}C_n}{{}^{2n}C_{n-1}} = \frac{15}{4}$$

$$\Rightarrow \quad \frac{(2n+2)(2n+1)}{n(n+2)} = \frac{15}{4}$$

$$\Rightarrow \quad n^2 - 6n + 8 = 0$$

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Chapter 15 – Probability

15 PROBABILITY

Exercise-1 : Single Choice Problems

2. $$f(x) = 3\sqrt{x} + 4\sqrt{1-x} \qquad \text{[where } x = P(A)\text{]}$$

$$f(x)_{\text{max.}} = 5 \text{ at } x = \frac{9}{25}$$

3. $$P(A \cup B) = 1 - P(\overline{A \cup B}) = \frac{5}{6}$$

$$P(A \cap B) = \frac{1}{4}, P(A) = \frac{3}{4}$$

$$\Rightarrow \quad P(B) = P(A \cup B) + P(A \cap B) - P(A) = \frac{1}{3}$$

4. $1 - \left(\frac{1}{2}\right)^n = \frac{31}{32} \Rightarrow n = 5$

5. Required probability $= \dfrac{3! \times 2}{\dfrac{9!}{3!3!3!}} = \dfrac{1}{140}$

6. Required probability $= \dfrac{\dfrac{3!(3n-3)!}{[(n-1)!]^3}}{\dfrac{(3n)!}{(n!)^3}}$

7. If product of two numbers equal to third number, then possibilities are (2, 3, 6), (2, 4, 8), (2, 5, 10).

Probability $= \dfrac{3}{{}^{10}C_3} = \dfrac{1}{40}$

8. $P = \frac{3}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$

9. Total word $= n = \frac{7!}{2!2!}$

T $\boxed{\text{ITANIC}}$ I $\boxed{\text{TTANIC}}$ A $\boxed{\text{TTINIC}}$

Favourable word $= m = \frac{6!}{2!} + \frac{6!}{2!} + \frac{6!}{2!2!} \Rightarrow P = \frac{m}{n} = \frac{5}{7}$

10. Probability $= \frac{n!}{n^n} = \frac{3}{32} = \frac{6}{64}$

$$\Rightarrow \quad \frac{(n-1)!}{n^{n-1}} = \frac{1 \cdot 2 \cdot 3}{4^3} \Rightarrow n = 4$$

11. Total case $= n = 9 \times 10^3$

Favourable case $= m = (9 \times 10^3) - 6^4$

$$P = 1 - \frac{6^4}{9 \times 10^3} = \frac{107}{125}$$

12. Total case $= n = 6!$

Favourable case $= m = (3! \times 2!) + (2! \times 2!) = 16$

Probability $= \frac{16}{6!} = \frac{1}{45}$

13. E_1 "No card is king from removed cards"

E_2 "1 card is king from removed cards"

E_3 "2 card is king from removed cards"

E_4 "3 card is king from removed cards"

E_5 "4 card is king from removed cards"

$F = 3$ cards are drawn from pack those are kings.

$$P(F) = \sum_{i=1}^{S} P(E_i) \cdot P\left(\frac{F}{E_i}\right) = \frac{{}^{48}C_{26}}{{}^{52}C_{26}} \cdot \frac{{}^{4}C_3}{{}^{26}C_3} + \frac{{}^{48}C_{25} \cdot {}^{4}C_1}{{}^{52}C_{26}} \cdot \frac{{}^{3}C_3}{{}^{26}C_3} + 0 + 0 + 0$$

$$= \frac{4}{{}^{52}C_{26} \cdot {}^{26}C_3}({}^{48}C_{26} + {}^{48}C_{25}) = \frac{4 \times {}^{49}C_{26}}{{}^{52}C_{26} \cdot {}^{26}C_3}$$

$$= \frac{1}{(13)(17)(25)}$$

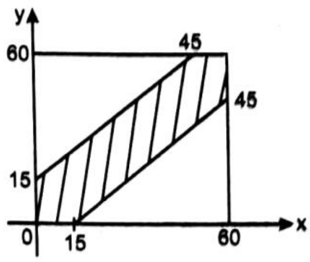
14. $\frac{{}^{3}C_2 \cdot {}^{10}C_4}{{}^{13}C_6} \times \frac{1}{7} = \frac{15}{286}$

15. Let f be function from $\{1, 2, \ldots, 10\}$ to itself total functions possible is 10^{10}. The number of one-one onto functions possible is 10!.

Hence, the probability of selected function to be one-one onto is $\frac{10!}{10^{10}} = \frac{9!}{10^9}$.

16. Let the friends come to the restaurant at 5 h x min and 5 h y min, respectively, where $x, y \in [0,60]$.

Hence, the sample space consists of all points (x, y) lying in 60×60 square as shown above and for favourable cases, $|x - y| \le 15$, that is $-15 \le x - y \le 15$ which is shown by shaded region in the graph shown below :



Hence, the probability that they will meet is given by :

$$1 - \frac{2 \times \frac{1}{2} \times 45 \times 45}{60 \times 60} = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$$

17. Total ways = ${}^{91}C_3$

Favourable ways = (Common ratio is 2) + (Common ratio is 3) = 16 + 2 = 18

Exercise-2 : One or More than One Answer is/are Correct

1. Probability $= \frac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8} \times \frac{1}{7} = \frac{8}{195}$

2. Probability $= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{8}\right) \ldots\ldots \left(1 - \frac{1}{2012}\right)$

$$= \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \ldots\ldots \times \frac{2011}{2012} = \frac{2012!}{2^{2012}(1006!)^2}$$

3. We have $P(E_i) = \frac{2}{4} = \frac{1}{2}$ or $i = 1, 2, 3$.

Also for $i \ne j, P(E_i \cap E_j) = \frac{1}{4} = P(E_j)P(E_i)$. Therefore, E_i and E_j are independent for $i \ne j$.

Also, $P(E_1 \cap E_2 \cap E_3) = \frac{1}{4} \ne P(E_1)P(E_2)P(E_3)$

$\therefore$ E_1, E_2, E_3 are not independent.

4. Max. $(P(A \cap B)) = P(A) = \frac{3}{5}$

$$\text{Min. } (P(A \cap B)) = P(A) + P(B) - 1 = \frac{4}{15}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{19}{15} - P(A \cap B)$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{3}{5} - P(A \cap B)$$

$$P\left(\frac{\bar{A}}{B}\right) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $P(E_1) = \frac{1}{10} \times 1 + \frac{2}{10} \times \frac{1}{2} + \frac{3}{10} \times \frac{1}{3} + \frac{4}{10} \times \frac{1}{4} = \frac{4}{10} = \frac{2}{5}$

2. $P\left(\frac{B_3}{E_2}\right) = \frac{P(B_3 \cap E_2)}{P(E_2)} = \frac{\frac{3}{10} \times \frac{1}{3}}{\frac{2}{10} \times \frac{1}{2} + \frac{3}{10} \times \frac{1}{3} + \frac{4}{10} \times \frac{1}{4}} = \frac{1}{3}$

Paragraph for Question Nos. 3 to 5

3. Mr. A's 3 digit number is always greater than Mr. B's 3 digit numbers then A should always pick digit 9.

$$\text{Probability} = \frac{{}^8C_3 \times {}^8C_2}{{}^8C_3 \times {}^9C_3} = \frac{1}{3}$$

4. $\text{Probability} = \frac{{}^8C_3 \times 1}{{}^9C_3 \times {}^8C_3} = \frac{1}{{}^9C_3} = \frac{1}{84}$

5. $P(E) = A$ picks 9 or A does not pick 9 and his number is greater than B

$$= \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}\left(1 - \frac{{}^8C_3}{{}^8C_3} \cdot \frac{1}{{}^8C_3}\right) = \frac{37}{56}$$

Paragraph for Question Nos. 6 to 7

6. Let a_n = number of ways of outcomes of n tosses when no 2 consecutive heads occur

$a_n = a_{n-2} + a_{n-1}$

Also, $a_1 = 2$ (H or T)

$a_2 = 3$ (TT or HT or TH)

$\therefore$ $a_3 = 5, a_4 = 8$ ......

$a_{10} = 144$

$\therefore$ Probability $= \dfrac{144}{2^{10}}$

7. [HT HT HTH] T, T, T

Number of ways of arranging $= \dfrac{4!}{3!} = 4$

$$\text{Probability} = \frac{4}{2^{10}}$$

Paragraph for Question Nos. 8 to 10

8. $6n > 2^n, n \in N$

$\therefore \quad n = 1, 2, 3, 4$

9. $\dfrac{4}{6} \times \left(\dfrac{\text{Number of solutions of } x + y > 4, 1 \le x, y \le 6}{36}\right)$

$$\times \left(\frac{\text{Number of solutions of } x + y + z > 8, 1 \le x, y, z \le 6}{6^3}\right)$$

$$= \frac{4}{6} \times \frac{30}{36} \times \frac{160}{216} = \frac{100}{243}$$

10. Probability $= \dfrac{4}{6} \times \dfrac{30}{36} \times \left(1 - \dfrac{160}{216}\right) = \dfrac{4}{6} \times \dfrac{30}{36} \times \dfrac{56}{216} = \dfrac{35}{243}$

Paragraph for Question Nos. 11 to 12

11. Let p_1 be the probability of being an answer correct from section 1. Then $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2. Then $p_2 = 1/15$.

Hence, the required probability is $\dfrac{1}{5} \times \dfrac{1}{15} = \dfrac{1}{75}$

12. Scoring 10 marks from four questions can be done in $3 + 3 + 3 + 1 = 10$ ways so as to answer 3 questions from section 2 and 1 question from section 1 correctly.

Hence, the required probability is $\dfrac{{}^{10}C_3 \; {}^{10}C_1}{{}^{20}C_4} \dfrac{1}{5}\left(\dfrac{1}{15}\right)^3$.

Exercise-5 : Subjective Type Problems

1. $\left(\dfrac{2}{3} + \dfrac{1}{3} \cdot \dfrac{2}{3} + \dfrac{1}{3} \cdot \dfrac{1}{3} \cdot \dfrac{2}{3}\right)\left(\dfrac{2}{3} + \dfrac{1}{3} \cdot \dfrac{2}{3}\right)\dfrac{2}{3} = \dfrac{416}{729}$

5. Probability $= \dfrac{{}^6C_5 + {}^7C_4 + {}^8C_3 + {}^9C_2 + {}^{10}C_1 + 1}{2^{10}} = \dfrac{9}{64}$

6. $p = \dfrac{^3C_1}{^7C_1} = \dfrac{3}{7}$

7. Total ways $= \dfrac{6!}{2!2!2!3!} \times 3! = 90$

Favourable cases $= 90 - [3! + {^3C_1} \times {^3C_1} \times 2 \times 2] = 48$

$\Rightarrow \quad p = \dfrac{48}{90} = \dfrac{8}{15}$

9. $E_1 \rightarrow$ be the event of both getting the correct answer

$E_2 \rightarrow$ both getting wrong answers.

$E \rightarrow$ both obtaining same answer.

$$P(E_1) = \frac{1}{8}\frac{1}{12} = \frac{1}{96}, \quad P(E_2) = \left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{12}\right) = \frac{77}{96}$$

$$P\left(\frac{E}{E_1}\right) = 1; \quad P\left(\frac{E}{E_2}\right) = \frac{1}{1001}$$

$$P\left(\frac{E_1}{E}\right) = \frac{1 \cdot \frac{1}{96}}{1 \cdot \frac{1}{96} + \frac{1}{1001} \cdot \frac{77}{96}} = \frac{13}{14}$$

10. Total ways $= {^9C_7} \times 7!$

Favourable ways $\Rightarrow {^9C_7} \times 7! - ({^7C_3} \times 3!) \times ({^6C_4} \times 4!)$

$$P(E) = 1 - \frac{({^7C_3} \times 3!) \times ({^6C_4} \times 4!)}{{^9C_7} \times 7!} = 1 - \frac{15}{36} = \frac{7}{12}$$

11. $\dfrac{1}{2}\left\{\dfrac{1}{2} \times \dfrac{1}{2} + 2 \times \dfrac{1}{4} \times \dfrac{1}{4}\right\} + \dfrac{1}{4} \times \left\{\left(\dfrac{1}{4} \times \dfrac{1}{4} \times \dfrac{1}{2}\right)^3\right\} = \dfrac{27}{128}$

12.

$1^{st}2^{nd}$	$\frac{1}{4} \times \frac{1}{6}$	a
$2^{nd}1^{st}$	$\frac{1}{4} \times \frac{1}{6}$	b
$1^{st}1^{st}$	$\frac{1}{4} \times \frac{1}{36}$	c
$2^{nd}2^{nd}$	$\frac{1}{4} \times \frac{1}{36}$	d

$\dfrac{c + d}{a + b + c + d} = \dfrac{2}{5}$

□□□

Chapter 16 – Logarithms

16 LOGARITHMS

Exercise-1 : Single Choice Problems

1. $\log_{10} x = A \qquad x > 0$

$\log_{10}(x-2) = B, \quad x - 2 > 0 \Rightarrow x > 2$

$\Rightarrow \qquad A^2 - 3AB + 2B^2 < 0$

$\Rightarrow \qquad (A - 2B)(A - B) < 0$

$\Rightarrow \qquad (\log x - 2\log(x-2))(\log x - \log(x-2)) < 0$

Case-I : $\quad \log x - 2\log(x-2) < 0$

and $\quad \log x - \log(x-2) > 0 \qquad \ldots(1)$

Case-II : $\quad \log x - 2\log(x-2) > 0$

and $\quad \log x - \log(x-2) < 0 \qquad \ldots(2)$

From (1) & (2), $\quad x \in (4, \infty)$

2. $(\log_e x)^2 = (\log_e x) - (\log_e x)^2 + 1 \qquad (\log_e x > 0)$

$2(\log_e x)^2 - \log_e x - 1 = 0$

$(2\log_e x + 1)(\log_e x - 1) = 0$

$\log_e x = -\frac{1}{2}$ (not possible)

$\log_e x = 1$

3. $S = (a^{\log_3 7})^{\log_3 7} + (b^{\log_7 11})^{\log_7 11} + (c^{\log_{11} 25})^{\log_{11} 25}$

$= 27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25} = 469$

4. $a^2 - 3a + 3 > \left(x + \frac{1}{x}\right)^0$ and $a^2 - 3a + 3 > 0$

$a^2 - 3a + 2 > 0$

$(a-1)(a-2) > 0 \quad \Rightarrow a \in (-\infty, 1) \cup (2, \infty)$

5. $P = \frac{5}{\log_x 120} = \log_{120} x^5; \quad (120)^P = x^5 = 32 \Rightarrow x = 2$

6. $x = \frac{z^{1/3}}{2}$, $y = \frac{z^{1/6}}{5}$

If $xy = z^{3/2}$; $\frac{z^{1/3}}{2} \cdot \frac{z^{1/6}}{5} = z^{3/2} \Rightarrow z = \frac{1}{10}$

7. $\log_x(\log_3(\log_x y)) = 0 \Rightarrow y = x^3, \log_y 27 = 1 \Rightarrow y = 27$

8. $\log_{10^{-2}} 10^3 + \log_{10^{-1}} 10^{-4} = \frac{-3}{2} + 4 = \frac{5}{2}$

9. $a = \frac{3}{1 + 2\log_3 2} \Rightarrow \log_3 2 = \frac{3-a}{2a}$; $\log_6 16 = \frac{4\log_3 2}{1 + \log_3 2}$

10. $\log_2(\log_2(\log_3 x)) = 0 \Rightarrow x = 9$

$\log_2(\log_3(\log_2 y)) = 0 \Rightarrow y = 8$

11. Let $\log_3 a = x$, $\log_3 b = y$; $\frac{x}{3} + \frac{y}{2} = \frac{7}{2}$ and $\frac{x}{2} + \frac{y}{3} = \frac{2}{3}$

12. $a = \log_2 5$; $b = \log_5 8$; $c = \log_8 11$; $d = \log_{11} 14$

$$2^{abcd} = 2^{\log_2 14} = 14$$

14. $\frac{\log_8 17}{\log_9 23} = \frac{\log_{2\sqrt{2}} 17}{\log_3 23}$

15. **Case-I :** $2x - 3 > 1$

$3x - 4 > 1$

$x > \frac{5}{3} \Rightarrow x > 2$

Case-II : $0 < 2x - 3 < 1$

$0 < 3x - 4 < 1$

$x < \frac{5}{3} \Rightarrow \frac{3}{2} < x < \frac{5}{3}$

16. $p \le \log_{10} N < p + 1 \quad \Rightarrow P = 10^{p+1} - 10^p$

$-q \le \log_{10} 1/N < -q + 1 \Rightarrow Q = 10^q - 10^{q-1}$

17. $n + 1$ = number of digits = 1 + characteristic

18. $\log_{10}(0.15)^{20} = 20(\log_{10} 15 - 2) = -16.478$

19. $\log_2(\log_4(\log_{10} 10^{16})) = \log_2(\log_4 16) = 1$

20. $2\log x - \log(2x - 75) = 2$

$$\frac{x^2}{2x - 75} = 100 \Rightarrow x^2 - 200x + 7500 = 0$$

21. $x^{\log_x a \cdot \log_a y \cdot \log_y z} = x^{\log_x z} = z$

22. $x^{x\sqrt{x}} = x^{3x/2}$

$x \ne 0, 1 \qquad x\sqrt{x} = \frac{3}{2}x \Rightarrow x = \frac{9}{4}$

If $x = 1$, then it also satisfy.

23. $(\log_3 x)^2 = 2\log_3 x$

$\Rightarrow \quad \log_3 x = 0 \quad \text{or} \quad \log_3 x = 2$

$x = 1 \quad \text{or} \quad x = 9$

24. $\log_{10} x + \log_{10} y = 2 \Rightarrow xy = 100$ ...(1)

$x - y = 15$ ...(2)

$\Rightarrow \quad x = 20, \quad y = 5$

25. $\left(2^{x+\frac{1}{3}\left(2x-\frac{3}{x}\right)}\right)^{1/2} = 2^{\frac{7}{3}}$

$\Rightarrow \quad x + \frac{2}{3}x - \frac{1}{x} = \frac{14}{3} \Rightarrow 5x^2 - 14x - 3 = 0$

26. $25^{(2x-x^2+1)} + 9^{(2x-x^2+1)} = 34\frac{3^{2x-x^2+1}}{3}\cdot\frac{5^{2x-x^2+1}}{5}$

Let $3^{2x-x^2+1} = a$ and $5^{2x-x^2+1} = b$

$$a^2 + b^2 = \frac{34}{15}ab$$

$$15a^2 - 34ab + 15b^2 = 0 \Rightarrow (3a-5b)(5a-3b) = 0$$

Case-1 : if $\frac{a}{b} = \frac{5}{3}$

$\Rightarrow \quad \left(\frac{3}{5}\right)^{2x-x^2+1} = \frac{5}{3}$

$\Rightarrow \quad 2x - x^2 + 1 = -1 \Rightarrow x^2 - 2x - 2 = 0$

Sum of two values of $x = 2$

Case-2 : if $\frac{a}{b} = \frac{3}{5}$

$$\left(\frac{3}{5}\right)^{2x-x^2+1} = \frac{3}{5}$$

$\Rightarrow \quad 2x - x^2 + 1 = 1 \Rightarrow x = 0$ and 2

Sum of all values of x is 4.

27. $a^x = b^y = c^z = d^w$

$\Rightarrow \quad b = a^{x/y},\ c = a^{x/z},\ d = a^{x/w}$

$$\log_a(bcd) = \log_a a^{\left(\frac{x}{y}+\frac{x}{z}+\frac{x}{w}\right)} = \frac{x}{y} + \frac{x}{z} + \frac{x}{w} = x\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right)$$

28. $x = \dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$

Multiply and divide by $(1-\sqrt[16]{5})$ then

$$x = -1 + \sqrt[16]{5}$$

$$(x+1)^{48} = 5^3 = 125$$

29. $$\log_x \log_{18}(\sqrt{2}+\sqrt{8}) = \frac{1}{3}$$

$$\log_x \log_{(3\sqrt{2})^2} 3\sqrt{2} = \frac{1}{3}$$

$$\log_x\left(\frac{1}{2}\right) = \frac{1}{3} \Rightarrow x = \frac{1}{8}$$

30. $f(n) = \frac{1}{3}\log_2 n$ if $\log_8 n$ is integer

$= 0$ otherwise

$$\sum_{n=1}^{2011} f(n) = \log_8 2^3 + \log_8 2^6 + \log_8 2^9 = 1+2+3 = 6$$

32. $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1) \Rightarrow (x-1)^2 > x-1$

$\Rightarrow (x-1)(x-2) > 0$

Also, for log to be defined $(x-1) > 0$

$$x \in (2, \infty)$$

33. $$\sqrt{7^{2x^2-5x-6}} = (49)^{3\log_2\sqrt{2}} = 7^3$$

$\Rightarrow \quad 2x^2 - 5x - 6 = 6$

$2x^2 - 5x - 12 = 0$

$\Rightarrow \quad (2x+3)(x-4) = 0$

34. $(\log_2 x)^4 + 16(\log_2 x)^2 \log_2 \dfrac{16}{x}$

$\Rightarrow \quad t^4 + 16t^2(4-t)$ (where $\log_2 x = t$)

$\Rightarrow \quad t^2(t^2 + 64 - 16t)$

$\Rightarrow \quad t^2(t-8)^2$

Since $1 \le x \le 256 \Rightarrow 0 \le t \le 8$

$\Rightarrow$ Maximum of $(t-8)^2 t^2$ lies at $t = 4$.

Hence, maximum $(4-8)^2 \cdot 4^2 = 256$

37. $\because \quad \lambda > 0$

$\therefore \quad \log_{16} x = \dfrac{1 \pm \sqrt{(1 - 4\log_{16}\lambda)}}{2}$

$\because$ The given equation will have exactly one solution, if

$1 - 4\log_{16}\lambda = 0 \quad \text{or} \quad \log_{16}\lambda = \dfrac{1}{4} = 4^{-1}$

$\therefore \quad \lambda = (16)^{4^{-1}} = (2^4)^{1/4} = 2, -2, 2i, -2i$, where $i = \sqrt{-1}$

But λ is real and positive.

$\therefore \quad \lambda = 2$

Number of real values = 1

38. Let x be the rational number, then according to question,

$$x = 50 \times \log_{10} x$$

By trial $x = 100$

39. $\because \quad x = \log_5(1000) = \log_5(5^3 \times 8) = 3 + \log_5 8$

and $y = \log_7(2058) = \log_7(7^3 \times 6) = 3 + \log_7 6$

$\Rightarrow \quad x - y = \log_5 8 - \log_7 6 > 0 \qquad \left(\begin{array}{l}\because \log_5 8 > 1, \log_7 6 < 1 \\ \therefore \log_5 8 - \log_7 6 > 0\end{array}\right)$

$\therefore \quad x > y$

40. $7\log\left(\dfrac{2^4}{5 \times 3}\right) + 5\log\left(\dfrac{5^2}{2^3 \times 3}\right) + 3\log\left(\dfrac{3^4}{2^4 \times 5}\right)$

$= 7\{4\log 2 - \log 5 - \log 3\} + 5\{2\log 5 - 3\log 2 - \log 4\} + 3\{4\log 3 - 4\log 2 - \log 5\}$

$= \log 2$

41. $\log_{10}\{\tan 1° \tan 2° \tan 3° \ldots \tan 45° \ldots \tan 87° \tan 88° \tan 89°\}$

$= \log_{10}\{\tan 1° \tan 2° \tan 3° \ldots \tan 45° \ldots \cot 3° \cot 2° \cot 1°\}$

$= \log_{10} 1 = 0$

42. $\log_7 \log_7 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = \log_7\left(\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8}\right) = \log_7\left(\dfrac{7}{8}\right)$

$= 1 - \log_7 8 = 1 - 3\log_7 2$

43. $(4)^{\log_{3^2} 3} + (9)^{\log_2 2^2} = (10)^{\log_x 83}$

$\Rightarrow \quad (4)^{1/2} + 9^2 = (10)^{\log_x 83}$

$\Rightarrow \quad (83)^1 = (83)^{\log_x 10}$

$\therefore \quad 1 = \log_x 10 \Rightarrow x = 10$

44. $(10^{\log_{10} x})^{\log_{10}\left(\frac{y}{z}\right)} (10^{\log_{10} y})^{\log_{10}\left(\frac{z}{x}\right)} (10^{\log_{10} z})^{\log_{10}\left(\frac{x}{y}\right)}$

45. $\because \quad \log_x 2 \log_{2x} 2 = \log_{4x} 2$

$\therefore \quad x > 0, 2x > 0$ and $4x > 0$ and $x \neq 1, 2x \neq 1, 4x \neq 1$

$\Rightarrow \quad x > 0$ and $x \neq 1, \frac{1}{2}, \frac{1}{4}$

Then, $\quad \frac{1}{\log_2 x} \cdot \frac{1}{\log_2 2x} = \frac{1}{\log_2 4x}$

$\Rightarrow \quad \log_2 x \cdot \log_2 2x = \log_2 4x$

$\Rightarrow \quad \log_2 x \cdot (1 + \log_2 x) = (2 + \log_2 x)$

$\Rightarrow \quad (\log_2 x)^2 = 2$

$\Rightarrow \quad \log_2 x = \pm\sqrt{2}$

$\therefore \quad x = 2^{\pm\sqrt{2}}$

$\therefore \quad x = \{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$

46. $\because \quad 2\log_{10} x - \log_x (0.01) = 2\log_{10} x - \log_x (10^{-2})$

$= 2(\log_{10} x + \log_x 10)$ $\quad (\because x > 0$ and $x \neq 1)$

$= 2\left(\frac{\log_e x}{\log_e 10} + \frac{\log_e 10}{\log_e x}\right) \geq 2 \cdot 2$ $\quad (\because \text{AM} \geq \text{GM})$

$= 4$

47. Let $\sqrt{\log_2 x} = a$

$a^2 - 2a + 1 \quad \Rightarrow a = 1$

if $\quad \sqrt{\log_2 x} = 1 \quad \Rightarrow x = 2$

48. $\log_e (e^2 x^{\ln x}) = \log_e x^3$

$2 + (\ln x)^2 = 3 \ln x$

Let $\ln x = a$

$a^2 - 3a + 2 = 0 \quad \Rightarrow (a-2)(a-1) = 0$

$\Rightarrow x_1 = e^2, x_2 = e$

49. $M = \text{antilog}_{32} 0.6 = (32)^{0.6} = 2^3 = 8$

$N = 49^1 \cdot 49^{-\log_7 2} + 5^{-\log_5 4}$

$= \frac{49}{4} + \frac{1}{4} = \frac{25}{2}$

50. $\log_2(\log_2(\log_3 x)) = 0 \quad \Rightarrow x = 9$

$\log_3(\log_3(\log_2 y)) = 0 \quad \Rightarrow y = 8$

51. $|\log_{1/2} 10 + |\log_4 625 - \log_2 5| = |\log_{1/2} 10 + \log_2 5| = 1$

52. $\log_3 2 = \dfrac{\log_5 2}{\log_5 3} = \dfrac{\left(\dfrac{1}{2a}\right)}{b - \dfrac{1}{2a}} = \dfrac{1}{2ab-1}$

55. $(x-3)^2 = 9 \Rightarrow x = 6$

57. $\log_a\left[\left(\dfrac{16}{15}\right)^7 \cdot \left(\dfrac{25}{24}\right)^5 \cdot \left(\dfrac{81}{80}\right)^3\right] = 8$

$\Rightarrow \quad \log_a 2 = 8 \quad \Rightarrow \quad a = 2^{1/8}$

58. $\log_{2^3}(2^7) - \log_{3^2}(3^{-1/2}) = \dfrac{7}{3} + \dfrac{1}{4} = \dfrac{31}{12}$

59. $\left(\dfrac{1}{\sqrt{27}}\right)^2 \cdot \left(\dfrac{1}{\sqrt{27}}\right)^{-\left(\frac{\log_5 16}{2\log_5 9}\right)} = \left(\dfrac{1}{27}\right)\left(\dfrac{1}{\sqrt{27}}\right)^{-\log_3 2}$

$= \left(\dfrac{1}{27}\right) \cdot 2^{-\log_3 \frac{1}{\sqrt{27}}} = \dfrac{2\sqrt{2}}{27}$

60. $\log_2 \dfrac{(x-1)(x+2)}{3x-1} = \log_2 4 \quad \Rightarrow \dfrac{(x-1)(x+2)}{3x-1} = 4$

$\Rightarrow \quad x^2 - 11x + 2 = 0$

61. $\log_{100} 10 = \dfrac{1}{2}$

$\log_2(\log_4 2) = \log_2 1/2 = -1$

$\log_4[\log_2(256)^2]^2 = \log_4 16^2 = 4$

$\log_4 8 = \log_{2^2} 2^3 = \dfrac{3}{2}$

62. $\lambda = \log_5(\log_5 3) \Rightarrow 5^\lambda = \log_5 3$

$3^{k+5^{-\lambda}} = 3^k \cdot 3^{5^{-\lambda}} = 3^k \cdot 3^{\log_3 5} = 5 \cdot 3^k$

63. $\log_{10} b^4 = 2\pi \cdot \log_{10} a^2$

$\dfrac{\log_{10} b}{\log_{10} a} = \log_a b = \pi$

64. $2^x = 3^y = 6^{-z} = k$ (let)

$x = \log_2 k,\ y = \log_3 k,\ z = -\log_6 k$

$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \log_k 2 + \log_k 3 - \log_k 6 = 0$

65. $(\sqrt{2}-1)^3 = 5\sqrt{2} - 7$

66. $1+\log_a b=\frac{1}{4} \Rightarrow \log_a b=-\frac{3}{4} \Rightarrow \dfrac{\frac{1}{3}-\frac{1}{2}\log_a b}{1+\log_a b}=\frac{17}{6}$

68. Let $\log_y x=t$

$5t^2-26t+5=0 \Rightarrow (5t-1)(t-5)=0$

Either $x=y^5$ or $y=x^5$

69. $1-\frac{1}{\log_3 x}=\frac{1}{\log_3 x-1} \Rightarrow (\log_3 x-1)^2=\log_3 x \Rightarrow (\log_3 x)^2-3\log_3 x+1=0$

70. $\log_2 x+\frac{1}{2}\log_2 y+\frac{1}{2}\log_2 z=2 \Rightarrow \quad x\sqrt{y}\sqrt{z}=4$

$\log_3 y+\frac{1}{2}\log_3 x+\frac{1}{2}\log_3 z=2 \Rightarrow \sqrt{x}\cdot y\cdot\sqrt{z}=9$

$\log_4 z+\frac{1}{2}\log_4 x+\frac{1}{2}\log_4 y=2 \Rightarrow \sqrt{x}\cdot\sqrt{y}\cdot z=16 \Rightarrow xyz=24$

71. $\left(\frac{1}{49}\right)\cdot 2^{\log_7 1/49}+7^{-\log_{1/5} 5}$

$\frac{1}{49}\times\frac{1}{4}+7$

72. $\log_2(3-x)-\log_2\frac{1}{\sqrt{2}}+\log_2(5-x)=\frac{1}{2}+\log_2(x+7)$

$\Rightarrow \quad \log_2(3-x)(5-x)=\log_2(x+7)$

$\Rightarrow \quad x^2-9x+8=0 \Rightarrow x=8$

73. $\log_5 x=\log_x 5 \Rightarrow x=5,\frac{1}{5}$

74. $|x-1|^{\log_3 x^2-2\log_x 9}=(x-7)^7$

either $x=2$ or $\log_3 x^2-2\log_x 9=7$

$(\log_3 x-4)(2\log_3 x+1)=0$

75. $9^{x-1}+7=4(3^{x-1}+1)$

Let $3^x=t$

$\frac{t^2}{9}+7=4\left(\frac{t}{3}+1\right) \Rightarrow t^2-12t+27=0$

$(t-3)(t-9)=0$

76. If $\alpha>1$

$\log_\alpha 10>\log_\alpha 3>\log_\alpha e>\log_\alpha 2$

$\Rightarrow \log_{10}\alpha<\log_3\alpha<\log_e\alpha<\log_2\alpha$

78. $\sum_{r=1}^{4} \log_4 2^r = \sum_{r=1}^{4} \frac{r}{2} = 5$

79. $\log_3 2 + \log_3 5 = \log_3 10$

$\log_3 9 < \log_3 10 < \log_3 27$

80. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\frac{k}{2}\left(a^3 + \frac{1}{a^3}\right) = \frac{3}{2}\left(a + \frac{1}{a}\right) - \frac{4}{8}\left(a + \frac{1}{a}\right)^3$$

$$= \frac{-1}{2}\left(a^3 + \frac{1}{a^3}\right)$$

$\Rightarrow \quad k = -1$

81. $2x - 3 > 0 \cap x^2 - 5x - 6 > 0 \cap 2x - 3 \neq 1$

$x > \frac{3}{2} \cap (x-6)(x+1) > 0 \cap x \neq 2$

$\Rightarrow \quad (6, \infty)$

Exercise-2 : One or More than One Answer is/are Correct

1.

$$6(\log x)^2 + \log x - 1 = 0$$

$$(3\log x - 1)(2\log x + 1) = 0$$

$$x = 10^{1/3} \quad \text{or} \quad x = 10^{-1/2}$$

3. $3(\log_{10} 2)x^2 - (1 - \log_{10} 2)x = 2\log_{10} 2 - x$

$$\log_{10} 2(3x^2 + x - 2) = 0$$

$$\log_{10} 2(x+1)(3x-2) = 0$$

Roots of this eq. are $x = -1, \frac{2}{3}$

Sum of coeff. $= 2\log_{10} 2$ (irrational)

Discriminant $= b^2 - 4ac = 25(\log_{10} 2)^2$ (irrational)

4. $A = \min.(x^2 - 2x + 7)\ \forall\ x \in R \quad \Rightarrow A = 6$

$B = \min.(x^2 - 2x + 7)\ \forall\ x \in [2, \infty) \quad \Rightarrow B = 7$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

1. If $\alpha_1 = 4$, then $3^4 \le N < 3^5$

If $\alpha_2 = 2$, then $5^2 \le N < 5^3$

$\Rightarrow$ $81 \le N < 125$

3. If $\alpha_1 = 5$, then $3^5 \le N < 3^6$

If $\alpha_2 = 3$, then $5^3 \le N < 5^4$

If $\alpha_3 = 2$, then $7^2 \le N < 7^3$

$\Rightarrow$ $243 \le N < 343$

Paragraph for Question Nos. 4 to 5

Sol. $|x^2 - y^2| = 221$

Paragraph for Question Nos. 6 to 7

Sol. $(1 + 4\log_{p^2}(2p))^2 + (1 + \log_2 p)^2 = (1 + \log_2 4p)^2$

Let $\log_2 p = t$

$$\left(1 + 2\left(\frac{1+t}{t}\right)\right)^2 + (1+t)^2 = (3+t)^2 \Rightarrow t = \log_2 p = 2$$

Exercise-4 : Matching Type Problems

1. (A) $a = 3((\sqrt{7}+1) - (\sqrt{7}-1)) = 6$

$b = \sqrt{1296} = 36$

(B) $a = (\sqrt{3}+1) - (\sqrt{3}-1) = 2$

$b = (3+\sqrt{2}) - (3-\sqrt{2}) = 2\sqrt{2}$

(C) $a = (\sqrt{2}+1)$, $b = (\sqrt{2}-1)$

(D) $a = 2+\sqrt{3}$, $b = 2-\sqrt{3}$

Exercise-5 : Subjective Type Problems

1. $N = 6^{\log_{10} 40} \cdot 6^{2\log_{10} 5} = 6^{\log_{10} 1000} = 6^3 = 216$

2. $\log_b(a^{\log_2 b}) = \log_a(b^{\log_2 b})$

$\Rightarrow \quad \log_b a = \log_a b \quad \Rightarrow \quad a = b$ or $a = \frac{1}{b}$ (not possible)

$\log_a (c - (b-a)^2) = 3 \quad \Rightarrow \quad c = a^3$

$\Rightarrow$ Minimum value of $c = 8$ at $a = 2$

3. $\log_b 729 = 6 \log_b 3$

if this is an integer, then $b = 3, 3^2, 3^3, 3^6$

4. **Case-1 :** If $\quad x + \frac{5}{2} > 1 \Rightarrow x > -\frac{3}{2}$

then $\quad (x-5)^2 < (2x-3)^2 \Rightarrow 3x^2 - 2x - 16 > 0 \Rightarrow x \in \left(\frac{8}{3}, \infty\right)$

Case-2 : If $\quad 0 < x + \frac{5}{2} < 1 \Rightarrow -\frac{5}{2} < x < -\frac{3}{2}$

then $\quad (x-5)^2 > (2x-3)^2 \Rightarrow x \in \left(-2, -\frac{3}{2}\right)$

there is no negative integral value of x.

5. $\frac{6}{5} a^{(\log_a x)(\log_{10} a)(\log_a 5)} - 3^{(\log_{10} x - 1)} = 9^{\left(\log_{100} x + \frac{1}{2}\right)}$

$$6 \cdot 5^{(\log_{10} x - 1)} - 3^{(\log_{10} x - 1)} = 3^{(\log_{10} x + 1)}$$

$$6 \cdot 5^{(\log_{10} x - 1)} = \frac{3^{\log_{10} x}}{3} + 3 \cdot 3^{\log_{10} x}$$

$$6 \cdot 5^{(\log_{10} x - 1)} = \frac{10}{3} \cdot 3^{\log_{10} x}$$

$$\left(\frac{5}{3}\right)^{\log_{10} x - 2} = 1$$

$\Rightarrow \quad \log_{10} x - 2 = 0$

$\Rightarrow \quad x = 100$

Integer part of $\log_3 100$ is 4.

6. $$\log_5\left(\frac{a+b}{3}\right) = \frac{\log_5 a + \log_5 b}{2}$$

$\Rightarrow \quad \log_5\left(\frac{a+b}{3}\right)^2 = \log_5 (ab)$

$\Rightarrow \quad (a+b)^2 = 9ab \Rightarrow a^2 - 7ab + b^2 = 0$

$$a^4 + b^4 + 2a^2b^2 = 49a^2b^2$$

$\Rightarrow \quad \frac{a^4 + b^4}{a^2b^2} = 47$

8. $\log_{10}\sqrt{1+x}+3\log_{10}\sqrt{1-x}=2+\log_{10}\sqrt{1-x}+\log_{10}\sqrt{1+x}$

$\Rightarrow \quad \log_{10}\sqrt{1-x}=1$

$\sqrt{1-x}=10 \Rightarrow x=-99$ (not possible)

9. $x^2=1+6\log_4 y$

$y^2-2^x y-2^{2x+1}=0$

$\Rightarrow \quad y=2^{x+1}$ and $y=-2^x$

if $\quad y=-2^x$ (not possible, because $y>0$)

if $\quad y=2^{x+1}$

$\Rightarrow \quad \log_2 y=x+1$

$x^2=1+3\log_2 y$

$\Rightarrow \quad x^2=1+3(x+1)$

$x^2-3x-4=0$

$\Rightarrow \quad (x-4)(x+1)=0$

$x_1=4$

$\Rightarrow \quad y_1=2^5=32$

$x_2=-1$

$\Rightarrow \quad y_2=2^{\circ}=1$

$\log_2|x_1x_2y_1y_2|=\log_2 128=7$

10. $\log_7\log_7\sqrt{7\sqrt{7\sqrt{7}}}=\log_7\log_7(7^{7/8})=\log_7(7/8)=1-3\log_7 2$

$\Rightarrow \quad a=3$

$\log_{15}\log_{15}\sqrt{15\sqrt{15\sqrt{15\sqrt{15}}}}=\log_{15}\log_{15}(15^{15/16})$

$=\log_{15}\left(\frac{15}{16}\right)=1-4\log_{15}2$

$\Rightarrow \quad b=4$

Then $\quad a+b=7$

11. $\log_{1+x}(1-y)+\log_{1-y}(1+x)=2 \qquad \left(t+\frac{1}{t}=2 \Rightarrow t=1\right)$

$1+x=1-y$

$x=-y$

$\therefore \quad \log_{1-y}(1+2y)+\log_{1-y}(1-2y)=2$

$$\log_{1-y}(1-4y^2)=2$$
$$1-4y^2=1+y^2-2y$$
$$5y^2-2y=0$$
$$y=0, y=\frac{2}{5}$$

But $y=0$ rejected.

12. $$\log_b n=2$$
$$\log_n(2b)=\log_n 2+\log_n b=2$$
$$\log_n 2+\frac{1}{2}=2$$
$$\log_n 2=\frac{3}{2} \Rightarrow n=2^{2/3}$$

if $\log_b n=2 \Rightarrow b=n^{1/2}=2^{1/3}$

$$n\cdot b=2^{2/3}\cdot 2^{1/3}=2$$

13. $$\log_y x+\frac{1}{\log_y x}=2$$

$\Rightarrow$ $\log_y x=1 \Rightarrow x=y$

$$x^2+y=12$$

$\Rightarrow$ $x^2+x-12=0$

$\Rightarrow$ $(x+4)(x-3)=0$

$\Rightarrow$ $x=-4$ or $x=3$

but $x>0$, then $x=3$

$$xy=9$$

14. $$y^x=x^y$$

if $x=2y$ then $y^{2y}=(2y)^y$

$\Rightarrow$ $2y\log y=y\log(2y)$

if $y\neq 0$ then $\log y^2=\log(2y)$

$\Rightarrow$ $y^2=2y \Rightarrow y=2$

$$x^2+y^2=5y^2=20$$

15. $(\log_2 4+\log_2(4^x+1))\log_2(4^x+1)=3$

Let $\log_2(4^x+1)=t$

$$t^2 + 2t - 3 = 0 \Rightarrow t = -3 \text{ or } 1$$

$$\log_2(4^x + 1) = 1 \Rightarrow 4^x = 1 \Rightarrow x = 0$$

17. $x^2 + 4x + 3 = 0 \quad (x > 0)$

18. $\log_{3^{1/4}}(\log_{3\sqrt{5}} x) = 4 \quad \Rightarrow \quad \log_3^{(\log_{3\sqrt{5}} x)} = 1$

$\Rightarrow \quad \log_5 x = 1 \quad \Rightarrow \quad x = 5$

❑❑❑

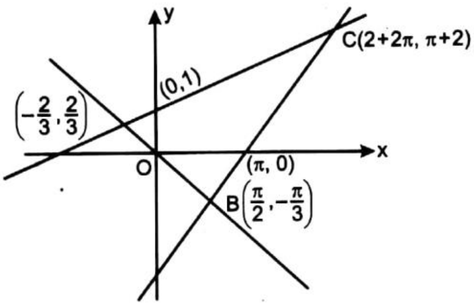
Chapter 17 – Straight Lines

17 STRAIGHT LINES

Exercise-1 : Single Choice Problems

1. Let ratio be $\lambda : 1 \Rightarrow \dfrac{6\lambda - 3}{\lambda + 1} = 0$, $\lambda = \dfrac{1}{2}$

3.



if $(a, \sin a)$ lie inside the triangle, then $a \in (0, \pi)$

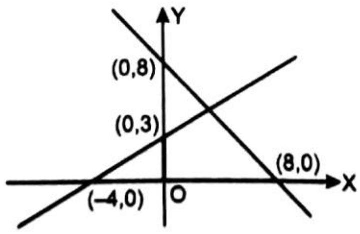
4. $$x = \frac{711}{13 + 11m} = \frac{9 \times 79}{13 + 11m}$$

if x is an integer, then $m = 6$

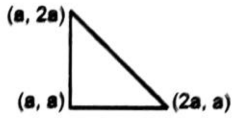
6. $$7\left(\frac{y}{x}\right)^2 + 2c\left(\frac{y}{x}\right) - 1 = 0$$

$$m_1 + m_2 = 4m_1 m_2 \Rightarrow c = 2$$

10.



13. $\frac{1}{2}a^2 = 72$



$a = \pm 12$

Centroid $\equiv (16, 16)$ or $(-16, -16)$

14. $g(x) = ax + b$

$g(1) = 2$

$\Rightarrow \quad a + b = 2$

$g(3) = 0$

$\Rightarrow \quad 2a = -2$

$a = -1$

$b = 3$

$g(x) = -x + 3$

$\cot[\cos^{-1}(|\sin x| + |\cos x|) - \sin^{-1}(|\sin x|) + |\cos x|]$

$|\sin x| + |\cos x| \in [1, \sqrt{2}]$

$\Rightarrow \quad \cot[\cos^{-1} 1 - \sin^{-1} 1] = 0 = g(3)$

15. Points A and B are mirror images about $y = x$.

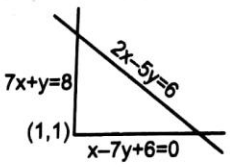
Point P will lie on the $\perp$ bisector of line joining A and $B \Rightarrow P$ lie on $y = x$.

16. $4m^3 - 3am^2 - 8a^2m + 8 = 0$ ⟨ m_1, m_2, m_3

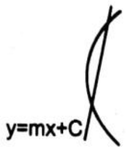
$m_1m_2m_3 = -2$

$\Rightarrow \quad m_3 = 2 \qquad (\because m_1m_2 = -1)$

17.



18. $2x^2 + 3y^2 - 5x\left(\frac{y - mx}{C}\right) = 0$



Coefficient of x^2 + coefficient of $y^2 = 0$

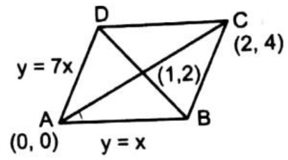
$5 + \frac{5m}{C} = 0 \Rightarrow m + C = 0$

Then the equation of family of line is $y = m(x - 1)$

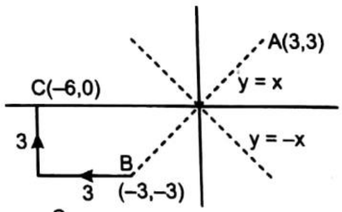
20. Equation of line BC is $y = 7x - 10$

Equation of line CD is $y = x + 2$

Area of rhombus $= \left|\frac{(2-0)(10-0)}{(7-1)}\right| = \frac{10}{3}$



21.

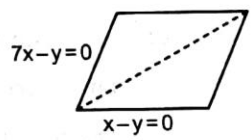


22. $y = \frac{3}{4}(x-9) + 6$

23. Acute angle bisector is

$$\frac{7x - y}{\sqrt{50}} = -\left(\frac{x - y}{\sqrt{2}}\right)$$

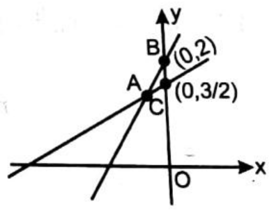
$\Rightarrow$ $y = 2x$



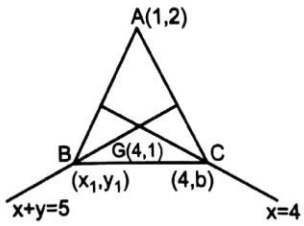
24. Either $x = y$ or $x = \left|\frac{3x + 4y - 12}{5}\right|$ or $y = \left|\frac{3x + 4y - 12}{5}\right| \Rightarrow (1, 1)$

25. Co-ordinate of point $A\left(-\frac{1}{7}, \frac{10}{7}\right)$

$\text{Ar}(\Delta ABC) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} = \frac{1}{28}$



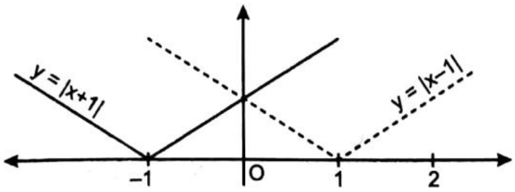
26.



Co-ordinate of centroid $G(4, 1) \Rightarrow \frac{x_1 + 4 + 1}{3} = 4$

$\Rightarrow$ $x_1 = 7$ and $y_1 = -2$

27.



The image of $y = |x-1|$ w.r.t. y-axis is $y = |x+1| \Rightarrow y = \pm(x+1)$

Required solution $= (y-(x+1))(y+(x+1)) = 0$

28.

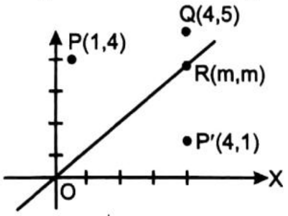
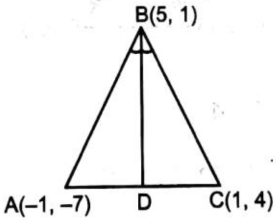


Image of $(1, 4)$ about the line $y = x$ is $(4, 1) \Rightarrow P'(4,1)\, Q(4,5)$ and $R(m,m)$ are collinear.

$\Rightarrow \qquad m = 4$

29. $\dfrac{AD}{CD} = \dfrac{AB}{BC} = \dfrac{10}{5} = \dfrac{2}{1}$



30. $4c\left(\dfrac{y}{x}\right)^2 - \left(\dfrac{y}{x}\right) + 6 = 0$ has one root is $-\dfrac{3}{4} \Rightarrow c = -3$

33. $$\frac{x}{a} + \frac{y(a+c)}{2ac} + \frac{1}{c} = 0$$

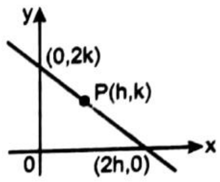
$$\Rightarrow \qquad a(y+2) + c(2x+y) = 0$$

Passes through a fixed point $(1, -2)$

34. $$\frac{1}{b}\left(\frac{y}{x}\right)^2 + \frac{2}{h}\left(\frac{y}{x}\right) + \frac{1}{a} = 0 \quad \begin{matrix} \nearrow m \\ \searrow 2m \end{matrix}$$

$$\Rightarrow \qquad 3m = -\frac{2b}{h} \text{ and } 2m^2 = \frac{b}{a} \Rightarrow \frac{ab}{h^2} = \frac{9}{8}$$

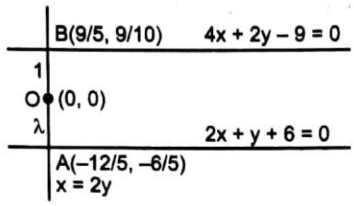
35. Equation of line is $\frac{x}{2h}+\frac{y}{2k}=1$



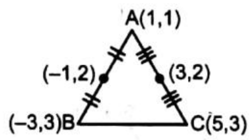
if it passes through fixed point (x_1, y_1)

$$\frac{x_1}{2h}+\frac{y_1}{2k}=1$$

36. $OA : OB = \lambda : 1 \Rightarrow \lambda = \frac{4}{3}$



37. $G\left(1, \frac{7}{3}\right)$



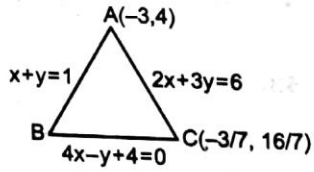
38. Diagonals are perpendicular.

39. Let point on the line $x + y = 4$ is $(a, 4-a)$.

$$\left|\frac{4(a)+3(4-a)-10}{5}\right|=1 \Rightarrow a^2+4a-21=0 \begin{cases} a_1 \\ a_2 \end{cases}$$

$\Rightarrow \quad a_1 + a_2 = -4 \Rightarrow b_1 + b_2 = 12$

40. Equation of altitude on BC



$$x + 4y = 13$$

Equation of altitude on AB

$$7x - 7y + 19 = 0$$

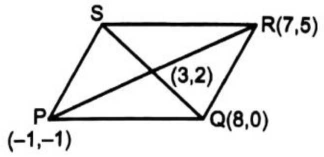
$\Rightarrow \quad H\left(\frac{3}{7}, \frac{22}{7}\right)$

41. Equation of line is $(3x+4y+5)+\lambda(4x+6y-6)=0$

$\Rightarrow \quad \frac{-(3+4\lambda)}{4+6\lambda}\times\frac{7}{5}=-1 \quad \Rightarrow \quad \lambda=\frac{1}{2}$

42. $\frac{5-1}{8-2}=\frac{7-5}{x-8} \Rightarrow x = 11$

43.



$\Rightarrow \quad S(-2, 4)$

44. Area $= \dfrac{1}{2}\begin{vmatrix} a & a & 1 \\ a+1 & a+1 & 1 \\ a+2 & a & 1 \end{vmatrix} = 1$

45. $(x-y)^2 = 1$

$\Rightarrow \quad x - y = 1$ and $x - y + 1 = 0$

46. AB subtend an acute angle at point C, then

$$a^2 + (a+1)^2 > 4$$

$$\Rightarrow \quad a \in \left(-\infty, \frac{-\sqrt{7}-1}{2}\right) \cup \left(\frac{\sqrt{7}-1}{2}, \infty\right)$$

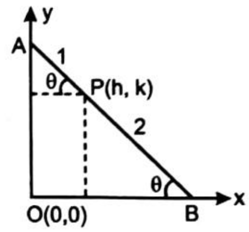
48.

$$h = \cos\theta$$

$$k = 2\sin\theta$$

$$h^2 + \frac{k^2}{4} = 1$$

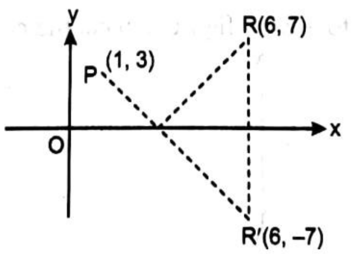
$$\Rightarrow \quad 4x^2 + y^2 = 4$$



50. Let the point of reflection is (h, k).

$$\frac{h-a}{1} = \frac{k-0}{-t} = \frac{-2(a+at^2)}{1+t^2} \Rightarrow x = -a$$

51.



52. Let (x, y) and (X, Y) be the old and the new coordinates, respectively. Since the axes are rotated in the anticlockwise direction, $\theta = +60°$. Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{X}{2} - \frac{\sqrt{3}}{2}Y \\ \frac{\sqrt{3}}{2}X + \frac{Y}{2} \end{bmatrix}$$

$$\Rightarrow \qquad x = \frac{X}{2} - \frac{\sqrt{3}}{2}Y \text{ and } y = \frac{\sqrt{3}}{2}X + \frac{Y}{2}$$

$$\Rightarrow \quad \left(\frac{X}{2} - \frac{\sqrt{3}}{2}Y\right)^2 - \left(\frac{\sqrt{3}}{2}X + \frac{Y}{2}\right)^2 = a^2$$

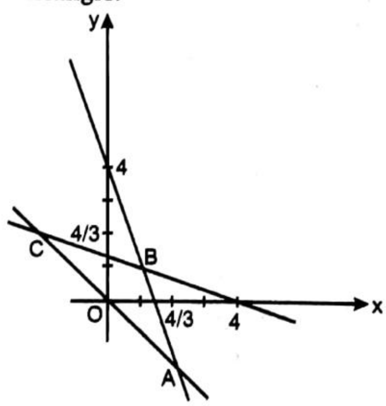
$$\Rightarrow \qquad (X^2 + 3Y^2 - 2\sqrt{3}XY) - (3X^2 + Y^2 + 2\sqrt{3}XY) = 4a^2$$

$$\Rightarrow \qquad -2X^2 + 2Y^2 - 4\sqrt{3}XY = 4a^2$$

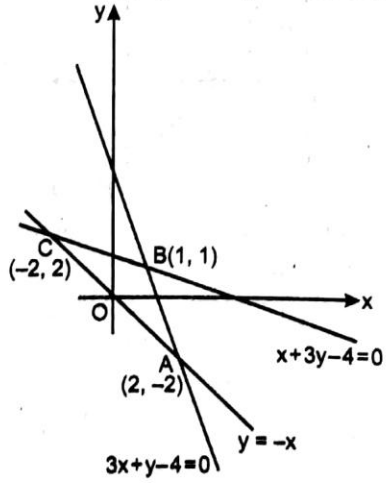
$$\Rightarrow \qquad Y^2 - X^2 - 2\sqrt{3}XY = 2a^2$$

which is the required equation.

53. The following figure depicts the condition. By observation from the figure, ΔABC is clearly an obtuse angled and isosceles triangle.



Alternate solution : The following figure depicts the condition.



From the figure, we get

$A: 3x + y = 4$ and $y = -x \Rightarrow x = 2; y = -2$

$B: (1, 1)$ by solving the equations.

$C: x + 3y - 4 = 0$ and $y = -x \Rightarrow x = -2; y = 2$

Thus, $AB = BC = \sqrt{1+9} = \sqrt{10}$

$$AC = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

$$\cos B = \frac{10 + 10 - 16(2)}{2(\sqrt{10})(\sqrt{10})} < 0$$

Therefore, the given triangle is isosceles and obtuse angled triangle.

56. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \Rightarrow$ Points are collinear.

57. $3h = a\cos t + b\sin t + 1$

$3k = a\sin t - b\cos t$

$\Rightarrow \quad (3h-1)^2 + (3k)^2 = (a\cos t + b\sin t)^2 + (a\sin t - b\cos t)^2 = a^2 + b^2$

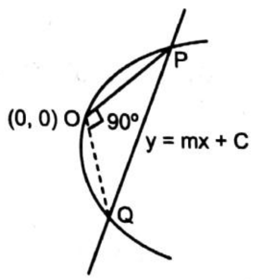
58. Equation of line $\frac{x}{a} + \frac{y}{-1-a} = 1.$

Lines passes from (4, 3).

62. The given triangle is equilateral. Therefore, the orthocentre of the triangle is same as centroid of the triangle. Thus, the orthocentre, that is, the centroid is given by

$$\left(\frac{5 + 0 + (5/2)}{3}, \frac{0 + 0 + (5\sqrt{3}/2)}{3}\right) \equiv \left(\frac{5}{2}, \frac{5}{2\sqrt{3}}\right)$$

63. Using homogenization,



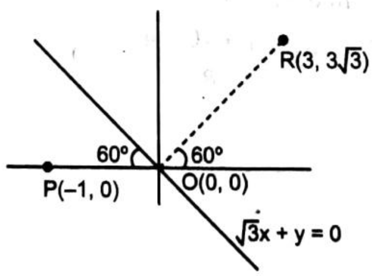
$$3x^2 - y^2 - 2x\left(\frac{y - mx}{C}\right) + 4y\left(\frac{y - mx}{C}\right) = 0$$

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\left(3 + \frac{2m}{C}\right) + \left(-1 + \frac{4}{C}\right) = 0$$

$$C = -m - 2$$

64.



Exercise-2 : One or More than One Answer is/are Correct

1. Let line be $\frac{x}{a}+\frac{y}{b}=1$

$$a+b=9 \quad \text{and} \quad ab=20$$

$\Rightarrow \quad a=5,\ b=4 \quad \text{or} \quad a=4,\ b=5$

2. Centroid is $\left(4, \frac{4}{3}\right)$.

3. $\begin{vmatrix} 2 & 3 & -5 \\ t^2 & t & -6 \\ 3 & -2 & -1 \end{vmatrix}=0 \Rightarrow t^2+t-6=0$

4.
$$b\left(\frac{y}{x}\right)^2+6\left(\frac{y}{x}\right)+a=0$$

$$bm^2+6m+a=0$$

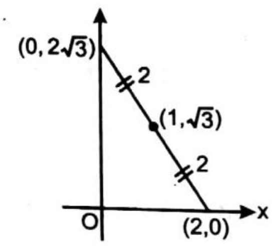
if $m=1$ is root of the equation

$\Rightarrow \quad a+b=-6$

if $m=-1$ is root of the equation

$\Rightarrow \quad a+b=6$

6. Co-ordinate of other two points



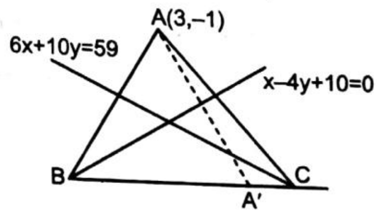
$$(1 \pm 2\cos\theta, \sqrt{3} \pm 2\sin\theta)$$

$$\left(1 \pm 2\left(\frac{\sqrt{3}}{2}\right), \sqrt{3} \pm 2\left(\frac{1}{2}\right)\right)$$

$$(1+\sqrt{3}, \sqrt{3}+1) \text{ and } (1-\sqrt{3}, \sqrt{3}-1)$$

8. Image of $A(3,-1)$ about angle bisector $x-4y+10=0$ is $A'(a,b)$.

$$\frac{a-3}{1}=\frac{b+1}{-4}=\frac{-2(3+4+10)}{17}$$



$\Rightarrow \quad A'(1,7)$

Let point $B\left(x_1, \frac{x_1+10}{4}\right)$ on the line $x-4y+10=0$

If mid-point of AB lie on the line $6x+10y=59$

$$6\left(\frac{x_1+3}{2}\right)+10\left(\frac{x_1+10-4}{8}\right)=59$$

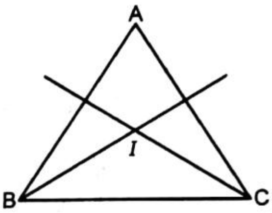
$\Rightarrow \quad B(10,5)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 3 to 4

3. $x + y = 2$ and $x - 3y = 6$

Meet at $(3, -1)$



4. Image of $A(2, -4)$ about $x + y = 2$ lie on BC.

$$\frac{x_2 - 2}{1} = \frac{y_2 + 4}{1} = -2\left(\frac{-4}{2}\right)$$

$\Rightarrow$ $\quad x_2 = 6, y_2 = 0$

Image of $A(2, -4)$ about $x - 3y = 6$ lie on BC.

$$\frac{x_3 - 2}{1} = \frac{y_3 + 4}{-3} = -2\frac{8}{10}$$

$\Rightarrow$ $\quad x_3 = \frac{2}{5}, y_3 = \frac{4}{5}$

Equ. of line BC, $\quad x + 7y = 6$

$\Rightarrow$ $\quad B\left(\frac{4}{3}, \frac{2}{3}\right)$ and $C(6, 0)$

Exercise-4 : Matching Type Problems

2. (A) $\sum_{r=1}^{n+1}({}^1C_{r-1} + {}^2C_{r-1} + {}^3C_{r-1} + \ldots + {}^nC_{r-1})$

$$= \sum_{r=1}^{n+1} {}^1C_{r-1} + \sum_{r=1}^{n+1} {}^2C_{r-1} + \sum_{r=1}^{n+1} {}^3C_{r-1} + \ldots + \sum_{r=1}^{n+1} {}^nC_{r-1}$$

$$= 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1)$$

(B) Family of line $(x + y + 2) + \lambda(2x - y + 4) = 0$ always passes from $(-2, 0)$.

If almost one tangent can be drawn from $(-2, 0)$ then

$$S_1 = 4 - 8g - 36 + 4g^2 \le 0$$

$$g^2 - 2g - 8 \le 0$$

(C) $2\sin 7x \cdot \cos 2x = \cos 2x$

$\Rightarrow$ $\quad \cos 2x = 0$ or $\sin 7x = \frac{1}{2}$

$$x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad 7x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}$$

(D) $a + b = \tan 65° \tan 70° - \tan 65° - \tan 70°$

$$\tan 135° = \frac{\tan 65° + \tan 70°}{1 - \tan 65° + \tan 70°} = -1$$

$\Rightarrow$ $\tan 65° \tan 70° - \tan 65° - \tan 70° = 1$

3. (A) $\cos 40° - 2\cos 40° \sin 10° = \cos 40° - (\sin 50° - \sin 30°)$

(B) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2\lambda & 4 \\ 1 & 1 & -3\lambda \end{vmatrix} = 0 \Rightarrow (3 - 2\lambda)(1 + 3\lambda) = 0$

(C) $\begin{vmatrix} k & 2-2k & 1 \\ -k+1 & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0 \Rightarrow 2k^2 + k - 1 = 0$

$\Rightarrow k = -1, \frac{1}{2}$

(D) $\sum_{k=3}^{\infty}\left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \ldots\ldots = \frac{1}{4}$

Exercise-5 : Subjective Type Problems

1. $\Delta = 132$

2. $ax + by + c = 3x - 4y + c$

$\Rightarrow$ $a = 3, b = -4$

Distance of $3x - 4y + c$ from $A(3, 1)$ is 1.

$\Rightarrow$ $\frac{|9 - 4 + c|}{5} = 1$

$|c + 5| = 5$

Also, $3x - 4y + c = 0$ and $3x - 4y + 5 = 0$ lie on same side of A

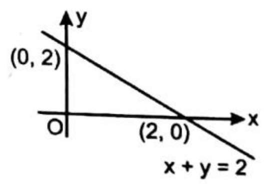
$\Rightarrow$ $c + 5 > 0$

$\Rightarrow$ $c + 5 = 5 \Rightarrow c = 0$

3. $xy(x + y - 2) = 0$

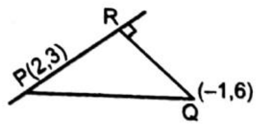
$\alpha + \alpha^4 - 2 \le 0 \qquad (\alpha > 0)$

$\Rightarrow$ $\alpha = 1$



5. $PQ = 3\sqrt{2}$

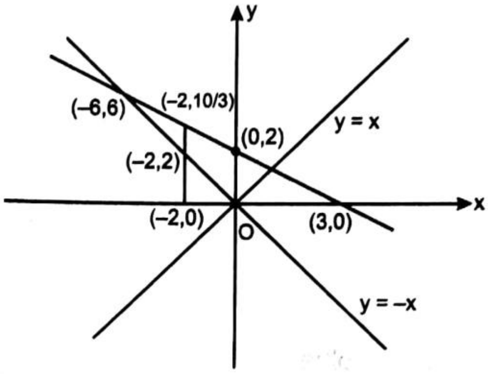
$QR \le PQ$



6. $x^2(y^2 - x^2) = 0$

has 3 different lines $x = 0, y = x$ and $y = -x$.

8. $2 < a < \frac{10}{3}$



9. Describe a circle whose diameter is AB.

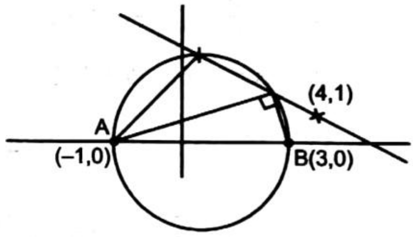
$\therefore$ centre $= (1, 0)$

Radius $= 2$

Let 'm' the slope of the line passing through $(4, 1)$.

$(y-1) = m(x-4)$ intersect the circle

$\perp$ distance from centre < radius of circle.



$$\left|\frac{-3m+1}{\sqrt{m^2+1}}\right| < 2$$

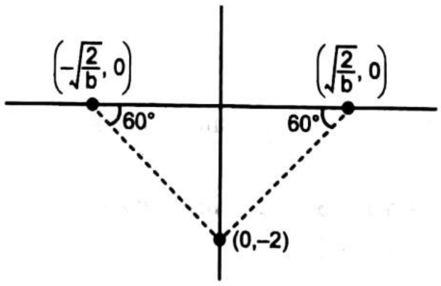
$$9m^2 - 6m + 1 < 4m^2 + 4$$

$$\Rightarrow \qquad m \in \left(\frac{6-\sqrt{96}}{10}, \frac{6+\sqrt{96}}{10}\right) - \left\{\frac{1}{5}, 1\right\}$$

$$\lambda_1 + \lambda_2 = \frac{12}{10} = \frac{6}{5}$$

$$5(\lambda_1 + \lambda_2) = 6$$

10. $\sqrt{\frac{2}{b}} = \frac{2}{\sqrt{3}} \Rightarrow b = \frac{3}{2}$



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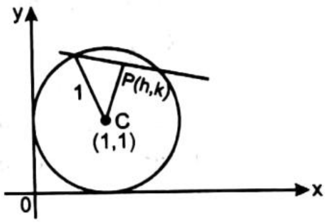
Chapter 18 – Circle

18 CIRCLE

Exercise-1 : Single Choice Problems

1. $CP = \frac{\sqrt{3}}{2} = \sqrt{(h-1)^2 + (k-1)^2}$



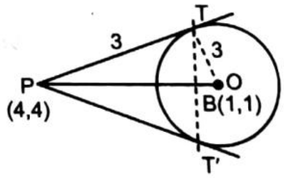
Locus of point $P(h, k)$ is

$$(x-1)^2 + (y-1)^2 = \frac{3}{4}$$

2. $\sqrt{d^2 - (r_1 - r_2)^2} = 15$; $\sqrt{d^2 - (r_1 + r_2)^2} = 5 \Rightarrow r_1 r_2 = 50$

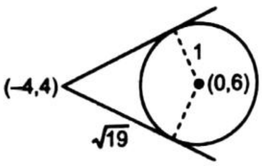
4. $$PT = \sqrt{16 + 16 - 8 - 8 - 7} = 3$$

$\Rightarrow \quad TT' = 2BT = 2 \cdot 3 \cos 45° = 3\sqrt{2}$



5. It will be circle with diametric ends as $(1, 1)$ and $(4, 2)$ *i.e.*, point of intersection.

6.



8. Let centroid be (h, k).

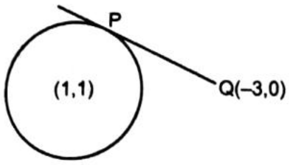
$\Rightarrow \quad h = \frac{\cos\alpha + \sin\alpha + 1}{3}, \quad k = \frac{\sin\alpha - \cos\alpha + 2}{3}$

$\Rightarrow \quad 3h - 1 = \cos\alpha + \sin\alpha, \quad 3k - 2 = \sin\alpha - \cos\alpha$

$\Rightarrow \quad (3h-1)^2 + (3k-2)^2 = 2$

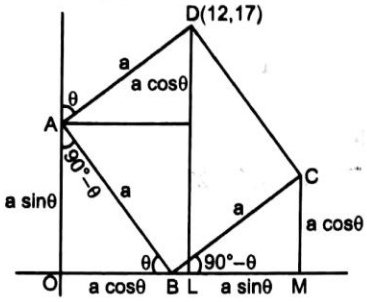
$\Rightarrow \quad \left(x - \frac{1}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$

9.



Length of tangent $= PQ = \sqrt{4^2 + 1^2 - 5} = \sqrt{12}$

10.

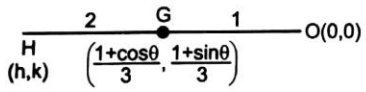


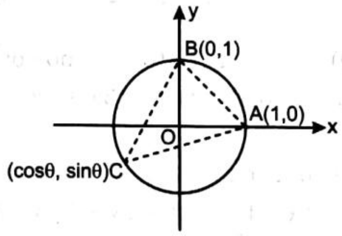
$OA = a\sin\theta = 12, \quad DL = a\sin\theta + a\cos\theta = 17$

$$a\cos\theta = 5$$

$$C = (a\cos\theta + a\sin\theta, a\cos\theta) = (17, 5)$$

12. Centroid divide the line joining orthocentre and circumcentre in 2 : 1.





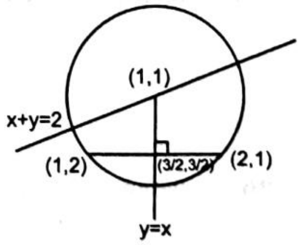
$\Rightarrow \quad h = 1 + \cos\theta, \quad k = 1 + \sin\theta$

$$(x-1)^2 + (y-1)^2 = 1$$

13. Co-ordinate of centre is $C(1, 1)$.

$$(x-1)^2 + (y-1)^2 = 1$$

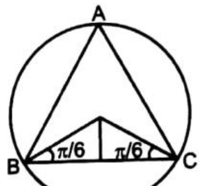
$$x^2 + y^2 - 2x - 2y + 1 = 0$$



14. $$a = 2R\cos\frac{\pi}{6}$$

$\Rightarrow$ $$a = 4\sqrt{3}\text{ cm}$$

$$\text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}a^2 = 12\sqrt{3}\text{ cm}^2$$



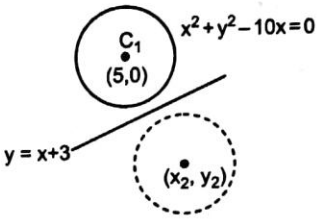
15. Image of centre $C_1(5,0)$ about the line $y = x + 3$ is

$$\frac{x_2 - 5}{1} = \frac{y_2 - 0}{-1} = \frac{-2(5+3)}{1^2 + 1^2}$$

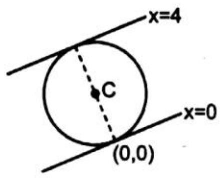
$\Rightarrow$ $C_2(-3, 8)$

Equation of reflected circle is

$$(x+3)^2 + (y-8)^2 = 25$$



19.



20. Let the equation of line is $3x + 4y = C$

$$\left|\frac{C}{5}\right| = 3 \Rightarrow C = 15 \text{ (in first quadrant)}$$

21. $C_1(5,0), C_2(3,-1), C_3(3/2, 2)$ do not lie on straight line.

22. Let equation of diameter is $3x + 5y = C$

$\Rightarrow$ $C = 7$

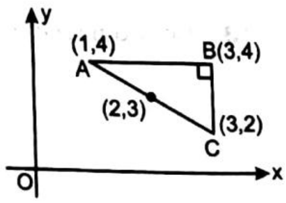
23. Equation of circle is

$$(x+1)(x-2) + (y-2)(y-3) + \lambda(x - 3y + 7) = 0$$

If its radius is $\sqrt{5}$.

$\Rightarrow$ $\lambda = \pm 1$

25. Equation of circle is $(x-1)(x-3) + (y-4)(y-2) = 0$

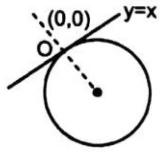


26. Equation of tangent at $O(0,0)$.

$$x(0) + y(0) + g(x+0) + f(y+0) = 0$$

$\Rightarrow$ $$gx + fy = 0$$

27. Equation of normal at $O(0,0)$



$$y=-x$$

$$\text{Centre}\left(0\pm\left(-\frac{1}{\sqrt{2}}\right),0\pm\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$\Rightarrow \quad \text{Either}\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\text{or}\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

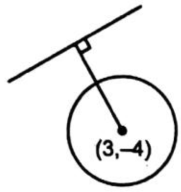
28. Here, $C_1C_2=r_1+r_2$ (Condition for external touch)

30. The triangle is right angled and the radical centre will be the orthocentre of the triangle.

32. Equation of common chord is $6x+14y+(l+m)=0$

If it passes through $(1,-4)$. Then, $l+m=50$

33. $$x^2+y^2-6x+8y=0$$

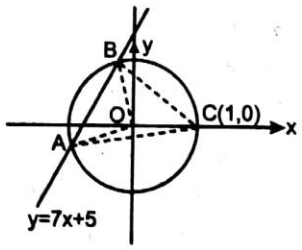


Distance of line from centre

$$\left|\frac{9-16-25}{5}\right|=\frac{32}{5}$$

$$\text{Shortest distance}=\frac{32}{5}-5=\frac{7}{5}$$

34. $\angle AOB=\frac{\pi}{2} \Rightarrow \angle ACB=\frac{\pi}{4}$



37. Equation of required circle :

$S:(x-1)^2+(y-1)^2+\lambda(x-y)=0$

$S':x^2+y^2+2y-3=0$

Common chord of $S=0$ and $S'=0$ is $S-S'=0$

$$(\lambda-2)x-(\lambda+4)y+5=0$$

Centre of $S':(0,-1)$ lies on common chord $\Rightarrow \lambda=-9$

$S:(x-1)^2+(y-1)^2-9(x-y)=0$

$$\Rightarrow \quad r=\frac{9}{\sqrt{2}}$$

40. Point lie inside the circle $k^2+(k+2)^2<4 \Rightarrow 2k^2+4k<0; -2<k<0$

41. The length of the normal is

$$\left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$$

The length of radius vector of a point (x, y) on the curve is $|xi + yi|$, that is $\sqrt{x^2 + y^2}$, it is given that

$$\sqrt{x^2 + y^2} = |y|\sqrt{1+(y')^2}$$

Squaring on both sides of this equation, we get

$$x^2 + y^2 = y^2[1+(y')^2]$$

$$\Rightarrow \quad x^2 + y^2 = y^2 + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad x^2 = \left(y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y\frac{dy}{dx} = x \text{ or } y = \frac{dy}{dx} = -x$$

Now,
$$y\frac{dy}{dx} = x$$

$$\Rightarrow \quad y\,dy = x\,dx$$

Integrating on both sides, we get

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow \quad x^2 - y^2 = 2c \text{ or } x^2 - y^2 = \text{constant}$$

This answer does not exist in the given options. So, consider the other alternative.

$$y\,dy = -x\,dx$$

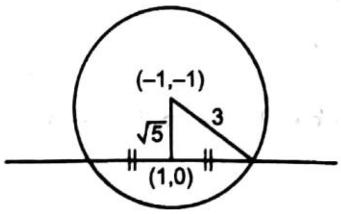
Integrating on both sides, we get

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow \quad x^2 + y^2 = \text{constant}$$

and this constant is > 0 in practical sense.

44. Length of chord $= 2\sqrt{3^2 - 5} = 4$



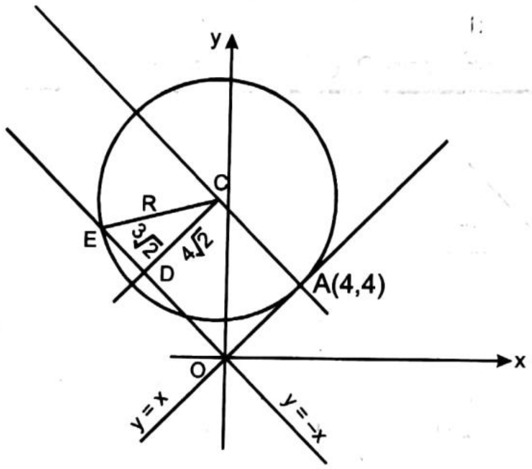
47. Family of circles touching the line $y = x$ at the point $(4, 4)$ is

$$(x-4)^2 + (y-4)^2 + \lambda(y-x) = 0$$

We need to find the member of this family which has length of chord $= 6\sqrt{2}$ on $x + y = 0$. For different λ's, we get different circles.

$$x^2 + y^2 - 8x - 8y + 32 + \lambda y - \lambda x = 0$$

$\Rightarrow$ $$x^2 + y^2 + x(-8-\lambda) + y(-8+\lambda) + 32 = 0 \quad \ldots(1)$$



Now,

$$OA = DC = 4\sqrt{2}$$

$$DE = 3\sqrt{2} = \frac{6\sqrt{2}}{2} \text{ (given)}$$

Therefore, $$R^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2$$

$\Rightarrow$ $$\frac{\lambda^2}{2} = 50 \Rightarrow \lambda^2 = 100 \Rightarrow \lambda = \pm 10$$

Substituting $\lambda = -10$ in eq. (1), we get

$$x^2 + y^2 + 2x - 18y + 32 = 0$$

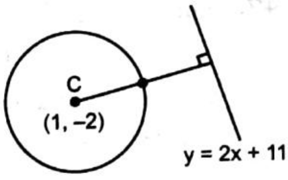
[Substituting $\lambda = 10$; in eq. (1); we get $x^2 + y^2 - 18x + 2y + 32 = 0$, which does not exist in the given options]

Note : From eq. (1), we get

$$R^2 = (\text{Radius})^2 = g^2 + f^2 - c = \frac{(\lambda+8)^2}{4} + \frac{(\lambda-8)^2}{4} - 32 = \frac{\lambda^2}{2}$$

48. Slope of line normal to circle and perpendicular to line

$$m = -\frac{1}{2} = \tan\theta$$

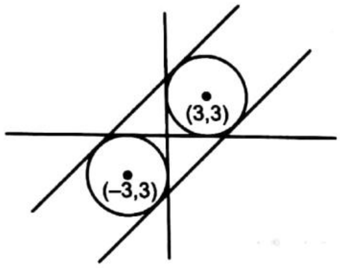


Co-ordinate of point lie on normal at a dist. of 3 from centre

$$\left(1 \pm 3\left(\frac{-2}{\sqrt{5}}\right), -2 \pm 3\left(\frac{1}{\sqrt{5}}\right)\right)$$

Exercise-2 : One or More than One Answer is/are Correct

2.

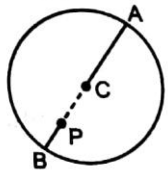


3. $x^2 + y^2 - x\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2} - 2\sin^{-1}\alpha\right) = 0$

$\Rightarrow$ Length of chord $= 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{4} - \sin^{-1}\alpha\right)^2}$

7. $(x+2)^2 + (y-3)^2$ is nothing but square of distance between (x, y) and $(-2, 3)$ where (x, y) is point lies on the circle.

Centre $= (-4, 5)$, $\quad r = \sqrt{16 + 25 + 40} = 9$



Clearly, $(-2, 3)$ is lies inside the circle.

$\therefore \quad PC = 2\sqrt{2} \qquad a = PA^2 = (9 + 2\sqrt{2})^2$

$b = PB^2 = (9 - 2\sqrt{2})^2$

$\therefore \quad a + b = 178, \quad a - b = 72\sqrt{2}$

8. Let point of intersection $P(h, k)$

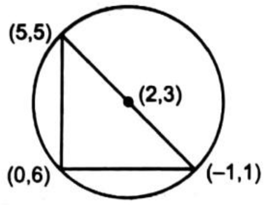
Equation of chord of contact is

$$hx + ky = a^2$$

If it is tangent to $x^2 + y^2 - 2ax = 0$

$$\Rightarrow \quad \left|\frac{ha - a^2}{\sqrt{h^2 + k^2}}\right| = a$$

9. Equation of tangent to circle



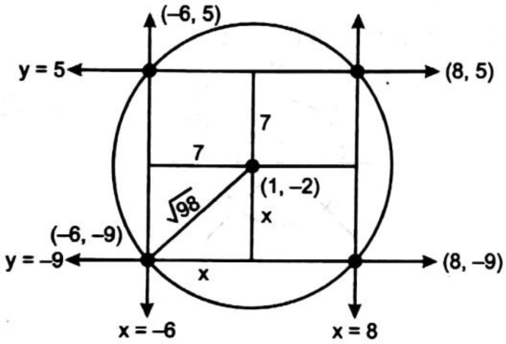
$$y-3=\frac{3}{2}(x-2)\pm\sqrt{13}\sqrt{1+\frac{9}{4}}$$

$\Rightarrow \; 2y=3x+13, \;\; 2y=3x-13$

$$\frac{x_2-2}{3}=\frac{y_2-3}{2}=-\left(\frac{13}{13}\right) \quad \Rightarrow (-1,5)$$

$$\frac{x_3-2}{3}=\frac{y_3-3}{-2}=-\left(\frac{-13}{13}\right) \quad \Rightarrow (5,1)$$

10. $2x^2=98 \Rightarrow x^2=49 \Rightarrow x=\pm 7$



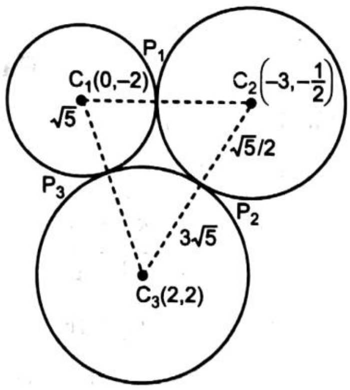
Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

Sol. $P_1(-2,-1)$

$P_2\left(-\frac{16}{7},-\frac{1}{7}\right)$

$P_3\left(\frac{1}{2},-1\right)$



Paragraph for Question Nos. 4 to 6

4. $S: x^2+y^2+x(2\lambda-9)+y(3\lambda-12)+53-27\lambda=0$

$C: x^2+y^2-4x-6y-3=0$

Equation of line : $S - C = 0$

or $x(2\lambda - 5) + y(3\lambda - 6) + 56 - 27\lambda = 0$

or $5x + 6y - 56 = 0$ or $2x + 3y - 27 = 0$

$\Rightarrow$ $x = 2, \quad y = \frac{23}{3}$

5. Centre of C lies on common chord of S and C.

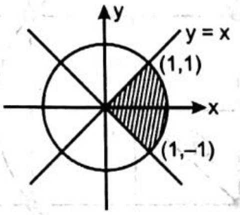
$\Rightarrow$ $(2, 3)$ lies on $x(2\lambda - 5) + y(3\lambda - 6) + 56 - 27\lambda = 0$

$\Rightarrow$ $S : x^2 + y^2 - 5x - 6y - 1 = 0$

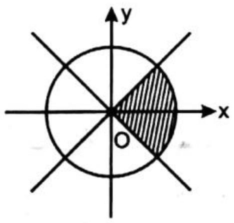
6. Difference of squares of lengths of tangents from A and B is 3, which is equal to $|AP^2 - BP^2|$.

Paragraph for Question Nos. 7 to 8

7. Max. dist. between any two arbitrary points = 2



8.



Paragraph for Question Nos. 9 to 10

Sol. Let $P(h, k)$

$L_1 = \sqrt{h^2 + k^2 - 4}$

$L_2 = \sqrt{h^2 + k^2 - 4h}$

$L_3 = \sqrt{h^2 + k^2 - 4k}$

If $L_1^4 = L_2^2 L_3^2 + 16$

$\Rightarrow$ $(h^2 + k^2 - 4)^2 = (h^2 + k^2 - 4h)(h^2 + k^2 - 4k) + 16$

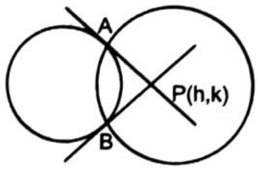
$\Rightarrow$ $(h + k)(h^2 + k^2 - 2h - 2k) = 0$

$C_1 : x + y = 0$

$C_2 : x^2 + y^2 - 2x - 2y = 0$

Exercise-5 : Subjective Type Problems

1. Equation of chord of contact w.r.t. P



$$hx + ky = 1$$

Equation of common chord is

$$(\lambda - 3)x + (2\lambda + 2)y + 3 = 0$$

$$\Rightarrow \quad \frac{\lambda - 3}{h} = \frac{2\lambda + 2}{k} = -3$$

$\Rightarrow$ Equation of locus is $6x - 3y - 8 = 0$

2. By using system of circles any circle passing through $(1, 1)$ and $(-2, 1)$ is

$$(x-1)(x+2) + (y-1)^2 + \lambda(y-1) = 0 \quad \text{...(1)}$$

Given circles $\quad x^2 + y^2 - 1 = 0 \quad$ ...(2)

Now radical axis of (1) and (2) is

$$(x - 2y) + \lambda(y - 1) = 0 \quad \text{...(3)}$$

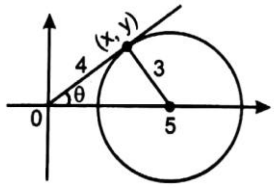
$\because$ Radical centre of given circles is $(0,0)$.

So, eq. (3) is passing through $(0, 0)$.

$\therefore \quad \lambda = 0$

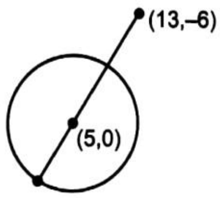
Put $\lambda = 0$ in eq. (1) we get required circle.

3. $\frac{y}{x} = \tan\theta = \frac{3}{4}$



4. $x^2 + y^2 - 26x + 12y + 210$

$(x - 13)^2 + (y + 6)^2 + 5$



5. $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow \quad 2g + 2f = -c - 2 \quad$ ...(1)

(1, 1) satisfy circle.

$\Rightarrow \quad 2g + 2f + c = -2$

$\Rightarrow \quad c = 0$

and $$g + f = -1$$

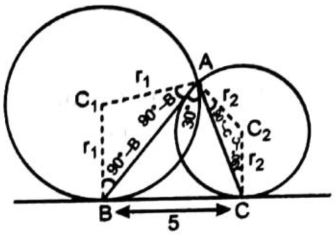
$\therefore$ Length of tangent $= \sqrt{8 + 4g + 4f + c} = 2$

6. Length of common external tangent

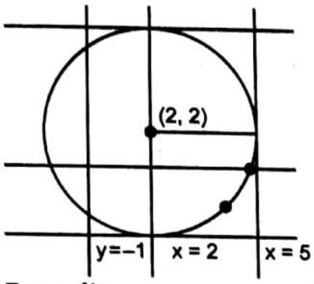
$$\sqrt{d^2 - (r_1 - r_2)^2} = 5 \quad \ldots(1)$$

$$\cos(90° - B + 90° - C + 30°) = \cos 60°$$

$$= \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2} \quad \ldots(2)$$



9.

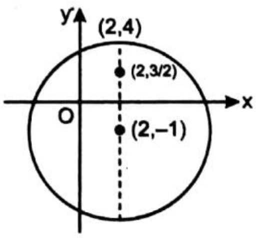


From diagram common points are 3.

10. $(C_1C_2)^2 = r_1^2 + r_2^2$

$$18 = 2r^2 \Rightarrow r^2 = 9$$

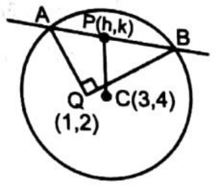
11.



12. $PQ = PA = PB$

$$\sqrt{(h-1)^2 + (k-2)^2} = \sqrt{6^2 - (h-3)^2 - (k-4)^2}$$

$$\Rightarrow \quad h^2 + k^2 - 4h - 6k - 3 = 0$$



13. $c = 3, a^2 + b^2 = 36$

Length of chord $AB = 2\sqrt{r^2 - p^2}$

$$= 2\sqrt{c - \left(\frac{2c}{\sqrt{a^2 + b^2}}\right)^2} = 2\sqrt{2}$$

□□□

Chapter 19 – Parabola

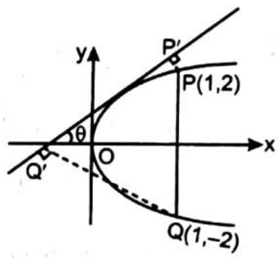
19 PARABOLA

Exercise-1 : Single Choice Problems

1. $P'Q' = PQ\cos(90° - \theta)$

$$= \frac{4}{\sqrt{t^2+1}} (t^2 < 1)$$

$(P'Q')_{min} = 2\sqrt{2}$



2. Equation of circle with SP as diameter

$$(x-4)\left(x-\frac{9}{4}\right)+y(y-6)=0$$

$$\text{Centre}\left(\frac{25}{8},3\right) \text{ and radius} = \frac{25}{8}$$

Equation of normal at $P(4,6)$ is

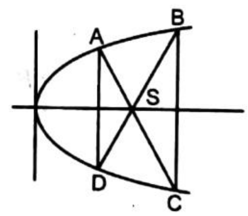
$$4x+3y=34$$

$$\text{Length of chord} = 2\sqrt{\left(\frac{25}{8}\right)^2-\left(\frac{\frac{25}{8}+9-34}{5}\right)^2} = \frac{15}{4}$$

3. The diagonals are the focal chord.

$$AS = 1+t^2 = c \text{ (say)}$$

$$\frac{1}{c}+\frac{1}{\left(\frac{25}{4}-c\right)}=1 \quad \left(\because \frac{1}{AS}+\frac{1}{CS}=\frac{1}{a}\right)$$



$\Rightarrow$ $$c=\frac{5}{4},5$$

$A\left(\frac{1}{4}, 1\right)$, $B(4, 4), C(4, -4)$ and $D\left(\frac{1}{4}, -1\right)$

Area of trapzium $= \frac{1}{2}(2+8) \times \frac{15}{4}$

4. For normal chord $t_2 = -t_1 - \frac{2}{t_1}$

Also chord substends an angle of 90° at the vertex

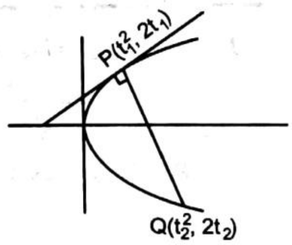
$\therefore \quad t_1 t_2 = -4 \quad \Rightarrow \quad t_2^2 = 8$

9. $(y - x + 2) + \lambda(y + x - 2) = 0$

The family of lines passes through $(2, 0)$.

The chord is $x = 2$ and end points are $(2, \pm 4)$.

10. $t_2 = -t_1 - \frac{2}{t_1}$



$h = \frac{t_1^2 + t_2^2}{2}$ and $k = \frac{2t_1 + 2t_2}{2}$

Put the value of t_2 and eliminate t_1 we get

$h - 2 = \frac{4}{k^2} + \frac{k^2}{2} \Rightarrow a = 2, b = 4, c = 2$

11. The parabola is $(y-1)^2 = 4(x-1)$. The coordinates of $P(1+t_1^2, 1+2t_1)$ and $Q(1+t_2^2, 1+2t_2)$.

Here $S(2, 1)$ is the focus. The coordinates of T are G.M. of abscissa and A.M. of ordinates of P and Q.

$\Rightarrow \quad ST^2 = 16 \qquad \therefore \; SP \cdot SQ = ST^2$

12. Let $P(t_1)$ and $Q(t_2)$ are point of $y^2 = 8x$

$2t_1^2 + 2t_2^2 = 17$ and $(2t_1^2)(2t_2^2) = 11$

$ST^2 = SP \cdot SQ = 2(1+t_1^2)\,2(1+t_2^2) = 34 + 4 + 11$

$ST = \sqrt{49}$

13. $ay = x^2 \Rightarrow \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-a}{2x_1} = -\frac{A}{B}$ (slope of normal)

$\Rightarrow x_1 = \frac{aB}{2A}$ and $y_1 = \frac{1}{B} - \frac{a}{2}$ put (x_1, y_1) in $ay = x^2$

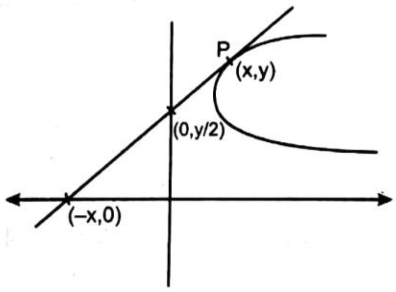
14.

$$\frac{dy}{dx}=\frac{y}{2x}$$

$$\frac{2dy}{y}=\frac{1}{x}dx$$

$$\Rightarrow \quad 2\log y=\log x+\log c$$

$$\Rightarrow \quad y^2=cx \quad \text{put } (3,1)$$



15.

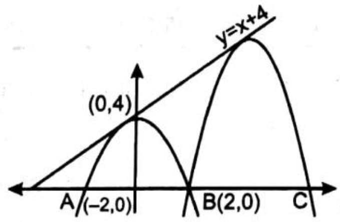
$$(x-\alpha)^2=-(y-(\alpha+4))$$

The curve passes through $(2,0)$

$$(2-\alpha)^2=-(0-(\alpha+4))$$

$$\alpha^2-5\alpha=0 \Rightarrow \alpha=0 \text{ or } \alpha=5$$

$$(x-5)^2=-(y-9) \quad \text{put } y=0 \Rightarrow x=2,8$$

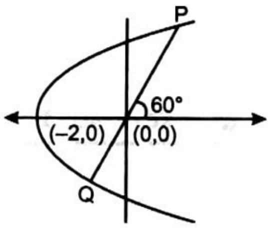


16. $y=(\tan 60°)\,x$ is the focal chord.

Coordinates of P and Q are intersection of $y=\sqrt{3}\,x$ with parabola

$$P(4,4\sqrt{3}),\ Q\left(-\frac{4}{3},\frac{-4}{\sqrt{3}}\right)$$

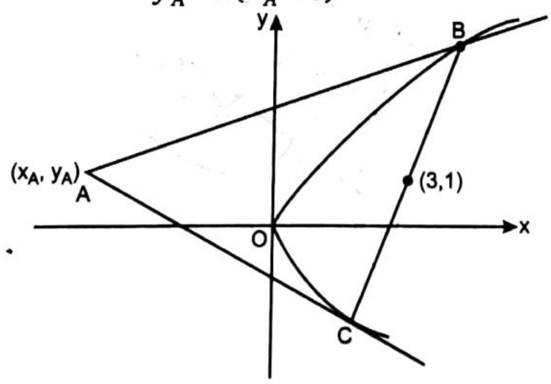
Find $\perp$ bisector of PQ.



17. The director circle of the parabola is its directrix $(x+11=0)$. Now apply condition of tangency.

18. The following figure depicts the condition. Chord of contact of a point $A(x_A, y_A)$ with respect to $y^2=4x$ is $y_A y=2(x+x_A)$. Since this chord passes through the point (3, 1), we have

$$y_A=2(x_A+3)$$



AB and *AC* are tangents to the parabola.

BC is chord of contact of point *A* with respect to the parabola $y^2=4ax$.

Given that point A lies on $x^2 + y^2 = 25$, we have

$$x_A^2 + y_A^2 = 25$$

$$\Rightarrow \quad x_A^2 + 4(x_A + 3)^2 = 25$$

$$\Rightarrow \quad x_A^2 + 4(x_A^2 + 9 + 6x_A) = 25$$

$$\Rightarrow \quad 5x_A^2 + 24x_A + 36 - 25 = 0$$

$$\Rightarrow \quad 5x_A^2 + 24x_A + 11 = 0$$

Exercise-2 : One or More than One Answer is/are Correct

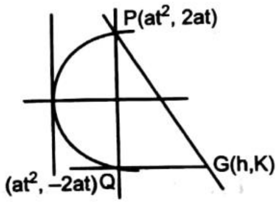
1. Equation of normal at $P(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

$$G(4a + at^2, -2at)$$

$\Rightarrow$ Locus of point $G(h, K)$ is

$$y^2 = 4a(x - 4a)$$



Exercise-3 : Comprehension Type Problems

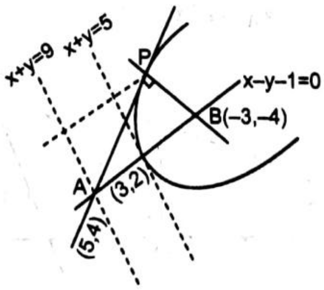
Paragraph for Question Nos. 1 to 3

Sol. Tangent and normal are angle bisectors of focal radius and perpendicular to directrix.

Circle 'C' circumscribing ΔABP is

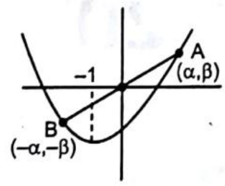
$$(x - 5)(x + 3) + (y - 4)(y + 4) = 0$$

Length of latus rectum $= 4(2\sqrt{2}) = 8\sqrt{2}$



Exercise-5 : Subjective Type Problems

1.



$$\beta = 2\alpha^2 + 4\alpha - 2 \quad \ldots(1)$$

$$-\beta = 2\alpha^2 - 4\alpha - 2 \quad \ldots(2)$$

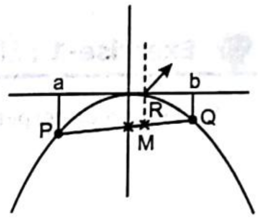
(1) & (2) $\Rightarrow 4\alpha^2 - 4 = 0 \Rightarrow \alpha = \pm 1$

Put $\alpha = 1, \beta = 2 + 4 - 2 = 4$

$\therefore$ $A(1, 4), B(-1, -4)$

$$AB^2 = l^2 = (\sqrt{4+64})^2 = 68$$

2. $R = \left(\dfrac{a+b}{2}, -\left(\dfrac{a+b}{2}\right)^2\right)$, $M = \left(\dfrac{a+b}{2}, \dfrac{-a^2-b^2}{2}\right)$



$$PQ = y + b^2 = \frac{-b^2 + a^2}{b - a}(x - b)$$

$$y + b^2 = -(b + a)(x - b)$$

$$y = -(b + a)(x - b) - b^2$$

$$\Delta_1 = \int_a^b [[-(a+b)(x-b) - b^2] + x^2]\,dx$$

$$= -(a+b)\frac{(x-b)^2}{2} - b^2x + \frac{x^3}{3}\Bigg|_a^b = \frac{(a-b)^3}{6}$$

$$\text{Area of } \Delta PQR = \Delta_2 = \frac{1}{2}\begin{vmatrix} a & -a^2 & 1 \\ b & -b^2 & 1 \\ \dfrac{a+b}{2} & -\left(\dfrac{a+b}{2}\right)^2 & 1 \end{vmatrix}$$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$, we get $\Delta_2 = \dfrac{(a-b)^3}{8}$

3.

$$m_{AB} \times m_{BC} = -1$$

$\Rightarrow$ $\dfrac{-2}{(t_1 + t_2)} \times \dfrac{-2}{(t_2 + t_3)} = -1$

$\Rightarrow$ $(t_1 + t_2)(t_2 + t_3) = -4$

Similarly,

$$m_{AD} \times m_{CD} = -1$$

$\Rightarrow$ $(t_1 + t_4)(t_3 + t_4) = -4$

$\Rightarrow$ $(t_1 + t_2)(t_2 + t_3) = (t_1 + t_4)(t_3 + t_4)$

Solving this

$$\frac{t_2 + t_4}{t_1 + t_3} = -1$$

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Chapter 20 – Ellipse

20 ELLIPSE

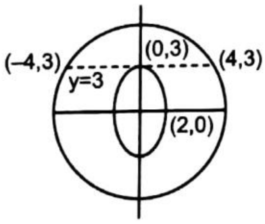
Exercise-1 : Single Choice Problems

1. Length of perpendicular from $C(0,0)$ to the tangent at $P(2\sqrt{3}\cos\theta, 2\sqrt{2}\cos\theta)$ is

$$CF = \left|\frac{-1}{\sqrt{\frac{\cos^2\theta}{12}+\frac{\sin^2\theta}{8}}}\right|$$

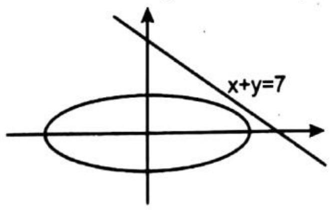
Equation of normal at P is $\frac{2\sqrt{3}x}{\cos\theta}-\frac{2\sqrt{2}\,y}{\sin\theta}=12-8$ which meets the major axis at $G\left(\frac{2}{\sqrt{3}}\cos\theta, 0\right)$

$$CF \times PG = 8$$

2. 

The minimum length of intercept will be possible when

$$y=3 \quad \text{or} \quad y=-3 \;\Rightarrow\; AB=8$$

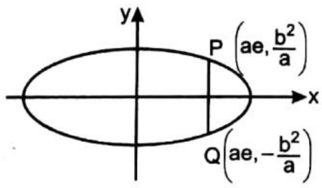
3. 

$$\frac{dy}{dx} = -\frac{x}{2y} = -1$$

Put $x = 2y$ in the equation of ellipse

The point lies in I quad $\Rightarrow (2, 1)$

4. Equation of tangent at P is



$$ex + y = a$$

$$\Rightarrow \quad e = \frac{2}{3}, \quad a = \frac{10}{3}$$

and $$b = \frac{10\sqrt{5}}{9}$$

Length of latus rectum $= \frac{2b^2}{a} = \frac{100}{27}$

5. Area bounded by circle & ellipse $= \pi a^2 - \pi ab = \pi a(a - b)$

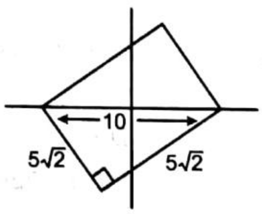
6. $\frac{S_1F_1 + S_2F_2}{2} \geq \sqrt{(S_1F_1)(S_2F_2)} = \sqrt{16}$

$\because$ Product of perpendiculars from two foci of an ellipse upon any tangent is equal to the square of semi-minor axis.

7. $$f(k^2 + 2k + 5) > f(k + 11)$$

$$\Rightarrow \quad k^2 + 2k + 5 < k + 11 \Rightarrow k \in (-3, 2)$$

8. Since sides of the square are tangent and perpendicular to each other, so the vertices lie on director circle



$$x^2 + y^2 = 16 + b^2 = \left(\frac{10}{2}\right)^2$$

$$\Rightarrow \quad b = 3$$

$$\frac{A}{\pi} = \frac{\pi(4)(3)}{\pi} = 12$$

9. $T = S_1 \quad \Rightarrow px + qy + \left(\frac{xq + py}{2}\right) - 1 = p^2 + q^2 + pq - 1$

$$\Rightarrow p^2 + q^2 = -pq \Rightarrow p = 0, q = 0$$

10. The combined equation of pair of tangents drawn from a point (x_1, y_1) to the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is $T^2 = SS_1$. Therefore,

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$$

$$\left(\frac{4x}{9}+2y-1\right)^2=\left(\frac{x^2}{9}+y^2-1\right)\left(\frac{4^2}{9}+2^2-1\right)$$

$\Rightarrow$ $$3x^2+7y^2-16xy+8x+36y-52=0$$

$\Rightarrow$ $$\tan\alpha=\frac{2\sqrt{h^2-ab}}{a+b}$$

where, $a=3, b=7$ and $h=-8$. Therefore,

$$\tan\alpha=\frac{2\sqrt{64-21}}{10}=\frac{\sqrt{43}}{5}$$

Note : α is acute angle between the pair of tangents. Therefore,

$$(a+b-c)^2=a^2+b^2+c^2+2ab-2ac-2bc$$

Alternate solution : Any line passing through the point (4, 2) is given by

$$y-2=m(x-4)$$

$$y=mx-4m+2$$

For this line to be tangent to the given ellipse, put this y into the equation of the ellipse and make

$$D=0$$

That is,

$$\frac{x^2}{9}+(mx-4m+2)^2=1$$

$$(1+9m^2)x^2+x(36m-72m^2)+16(9)m^2-16(9)m+27=0$$

Now,

$$D=0 \Rightarrow B^2-4AC=0$$

$\Rightarrow$ $$(36m-72m^2)^2-4(1+9m^2)(16\cdot 9m^2-16\cdot 9m+27)=0$$

$\Rightarrow$ $$(36m)^2(1-2m)^2-36(1+9m^2)(16m^2-16m+3)=0$$

$\Rightarrow$ $$m^2(1+4m^2-4m)-36(16m^2-16m+3+9\cdot 16m^4-9\cdot 16m+27m^2)=0$$

$\Rightarrow$ $$7m^2-16m+3=0$$

Now,

$$\tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}\right|$$

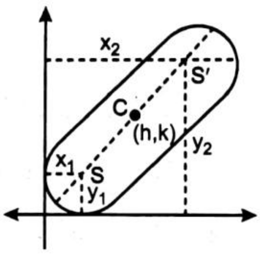
$\Rightarrow$ $$\tan\alpha=\frac{\sqrt{\left(\frac{16}{7}\right)^2-4\cdot\frac{3}{2}}}{1+\frac{3}{7}}=\frac{7}{10}\left(\frac{\sqrt{16^2-4\cdot 3\cdot 7}}{7}\right)$$

$$=\left(\frac{1}{10}\right)\sqrt{4(43)}=\frac{\sqrt{43}}{5}$$

where α is the acute angle between the tangents.

Exercise-2 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2



1. $SS'=2ae$

$$(x_1-x_2)^2+(y_1-y_2)^2=4(2)^2\left(\frac{\sqrt{3}}{2}\right)^2$$

$$(x_1+x_2)^2+(y_1+y_2)^2-4(x_1x_2+y_1y_2)=12$$

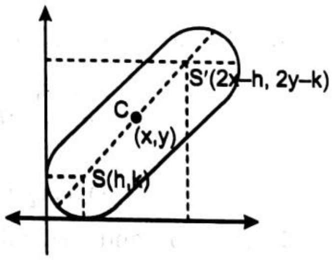
$$(2h)^2+(2k)^2-4(1+1)=12$$

($\because$ x_1x_2 and y_1y_2 are $\perp$ distance of the foci from their tangents $=b^2=1^2$)

$\Rightarrow$ $h^2+k^2=5$

2. $(2x-h)(h)=1 \quad\Rightarrow\quad x=\dfrac{1+h^2}{2h}$

$(2y-k)(k)=1 \quad\Rightarrow\quad y=\dfrac{1+k^2}{2k}$



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Chapter 21 – Hyperbola

21 HYPERBOLA

Exercise-1 : Single Choice Problems

1. The normal is $y-4=\frac{1}{4}(x-1)$. Put the value of y in $xy=4$ we get co-ordinates.

3. $c^2=a^2m^2-b^2 \Rightarrow c^2=\lambda^2m^2-(\lambda^3+\lambda^2+\lambda)^2$

$c^2 \geq 0 \quad \Rightarrow \quad m^2 \geq (\lambda^2+\lambda+1)^2$

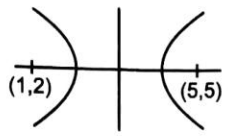
$\lambda^2+\lambda+1$ has minimum value $\frac{3}{4} \Rightarrow m^2 \geq \frac{9}{16}$

4. The asymptotes are $y=\pm\frac{\sqrt{3}}{2}x$ and the double ordinate be

$$P\left(h, \frac{\sqrt{3}}{2}\sqrt{h^2-4}\right) \text{ and } P'\left(h, -\frac{\sqrt{3}}{2}\sqrt{h^2-4}\right)$$

$\Rightarrow \quad (PQ)(PQ')=3$

5. $2ae=5$ and $2a=3$



$\Rightarrow \quad e=\frac{5}{3}$

$\Rightarrow \quad \frac{1}{e^2}+\frac{1}{(e')^2}=1 \Rightarrow e'=\frac{5}{4}$

6. The equation of normal *at* $(2\sec\theta, \tan\theta)$ is $2x\cos\theta+y\cot\theta=5$

Equal intercepts $\Rightarrow \sin\theta=\frac{1}{2}$

Also touches ellipse $\Rightarrow a^2+b^2=\frac{25}{3} \quad \because \; c^2=a^2m^2+b^2$

7. Let locus of point be (h, k).

Equation of chord of contact is $hx+ky=4$

For tangent, $x\left(\frac{4-hx}{k}\right)=1$ has two equal roots.

$\Rightarrow \quad hk = 4 \Rightarrow xy = 4$

8. $\frac{x^2}{16} - \frac{y^2}{18} - \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$

$\Rightarrow$ Coeff. of x^2 + coeff. of $y^2 = 0 \Rightarrow P = \pm 12$

The chord $x\cos\alpha + y\sin\alpha \pm 12 = 0$ is tangent to the circle $x^2 + y^2 = \left(\frac{d}{2}\right)^2 \Rightarrow \frac{d}{4} = 6$

9. Let the rectangular hyperbola be $x^2 - y^2 = a^2$ and the point be $(a\sec\theta, a\tan\theta)$.

$$a_1a_2 + b_1b_2 = (a\cos\theta)\left(\frac{2a}{\cos\theta}\right) + \left(-\frac{a\cos\theta}{\sin\theta}\right)\left(\frac{2a\sin\theta}{\cos\theta}\right)$$

Exercise-2 : One or More than One Answer is/are Correct

3. Let $\left(t, \frac{1}{t}\right)$ be any point on $xy = 1$

$$xy = 1$$

$\Rightarrow \quad xy' + y = 0$

$\Rightarrow \quad y' = \frac{-y}{x}$

$\Rightarrow \quad y' = -\frac{1}{t^2}$

$\Rightarrow \quad \frac{-b}{a} = t^2$

$\Rightarrow$ a and b are of opp. sign.

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Chapter 22 – Compound Angles

22 Compound Angles

Exercise-1 : Single Choice Problems

2. $a\sin x + b(2\cos c\cos x) = \alpha$

$$\cos c = \frac{\alpha - a\sin x}{2b\cos x}$$

$$= \frac{1}{2b}(\alpha\sec x - a\tan x) \text{ differentiate w.r.t. } x$$

$$\alpha\sec x\tan x - a\sec^2 x = 0$$

$$\Rightarrow \quad \sin x = \frac{a}{\alpha}$$

3. $$\tan x\cdot\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} < -1$$

$$t\left(\frac{3t - t^3}{1 - 3t^2}\right) + 1 < 0 \qquad (\text{Let } \tan x = t)$$

$$\frac{1 - t^4}{1 - 3t^2} < 0 \quad \Rightarrow \quad \frac{(t-1)(t+1)}{(3t^2 - 1)} < 0$$

$$\Rightarrow \quad t \in \left(\frac{1}{\sqrt{3}}, 1\right)$$

4. $$\sum_{r=1}^{8}\tan(rA)\tan\{(r+1)A\} = \sum_{r=1}^{8}\left[\frac{\tan(r+1)A - \tan(rA) - \tan A}{\tan A}\right] = \frac{\tan 9A - 9\tan A}{\tan A} = -10$$

5. $$f(x) = 2\operatorname{cosec} 2x + \sec x + \operatorname{cosec} x$$

$$= \frac{1 + \sin x + \cos x}{\sin x\cos x}$$

$$f'(x) = \frac{\sin^3 x + \sin^2 x - \cos^3 x - \cos^2 x}{\sin^2 x\cos^2 x} = 0 \quad \Rightarrow \quad x = \frac{\pi}{4}$$

$$f(x)_{\min} = \frac{2}{\sqrt{2}-1} \quad \text{at } x = \frac{\pi}{4}$$

9. cosec θ + cosec (60°−θ) − cosec (60°+θ)

where θ = 10°

10. $\frac{1}{2}(2\sin x\cos x + 2\cos^2 x) = \frac{1}{2}(\sin 2x + \cos 2x + 1)$

11. $\frac{\tan A}{\sqrt{3}} = \frac{\tan B}{\sqrt{5}} = k \qquad (k > 0)$, if $2\sin A = \sqrt{3}\sin B$

$$\Rightarrow \frac{2\tan A}{\sqrt{1+\tan^2 A}} = \frac{\sqrt{3}\tan B}{\sqrt{1+\tan^2 B}} \Rightarrow \frac{2\sqrt{3}k}{\sqrt{1+3k^2}} = \frac{\sqrt{3}\times\sqrt{5}k}{\sqrt{1+5k^2}} \Rightarrow k = \frac{1}{\sqrt{5}}$$

12. Gives equations can be written as

$$2\cos\alpha + 9\cos\delta = -6\cos\beta - 7\cos\gamma \qquad \ldots(1)$$

$$2\sin\alpha - 9\sin\delta = 6\sin\beta - 7\sin\gamma \qquad \ldots(2)$$

Square and add equation (1) and (2),

$$\Rightarrow 4 + 36 + 36[\cos\alpha\cos\delta - \sin\delta\sin\alpha] = 36 + 49 + 84[\cos\beta\cos\gamma - \sin\beta\sin\gamma]$$

$$\Rightarrow 36[\cos(\alpha+\delta)] = 84[\cos(\beta+\gamma)]$$

$$\frac{\cos(\alpha+\delta)}{\cos(\beta+\gamma)} = \frac{84}{36} = \frac{7}{3} = \frac{m}{n}; \qquad m + n = 10$$

13. $\left|\frac{1+\sin\theta+1-\sin\theta}{\sqrt{1-\sin^2\theta}}\right| = \left|\frac{2}{\cos\theta}\right| = -2\sec\theta$

14. $A = \sum_{r=1}^{3}\cos\frac{2r\pi}{7} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = B$

15.

$$\tan\beta = \frac{x}{z} = \frac{1}{3}$$

$$\tan\alpha = \frac{y}{z} = \frac{1}{2}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1 \quad \Rightarrow \quad \alpha + \beta = \frac{\pi}{4}$$

17. $f(x) = -2\sin^2 x + \sin x + 2 \ \forall \ x \in \left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$

Let $\sin x = t$

$f(t) = -2t^2 + t + 2 \ \forall \ t \in \left[\frac{1}{2}, 1\right]$

18. $1+(\cos^2 A-\sin^2 B)-\cos A\cos B = 1+\cos(A+B)\cos(A-B)-\frac{1}{2}[\cos(A+B)+\cos(A-B)]=\frac{3}{4}$

19. $(2\sin x-\text{cosec } x)^2+(\tan x-\cot x)^2=0$

$\therefore \quad \sin^2 x=\frac{1}{2} \cap \tan^2 x=1$

20. $\cos^2 A=\sin A\cdot\tan A \quad \Rightarrow \quad \cos^3 A=\sin^2 A$

21. $f(x)=\left(\frac{\sqrt{3}+1}{2}\right)\sin x+\left(\frac{\sqrt{3}+1}{2}\right)\cos x=\left(\frac{\sqrt{3}+1}{2}\right)(\sin x+\cos x)$

22. $A=B+C$

$\Rightarrow \quad \tan A\tan B\tan C=\tan A-\tan B-\tan C$

23.
$$E=\sin A+\sin 2B+\sin 3C$$
$$E=\frac{3}{5}+2\cdot\frac{4}{5}\cdot\frac{3}{5}-1$$
$$=\frac{15}{25}+\frac{24}{25}-1=\frac{39-25}{25}=\frac{14}{25}$$

24. $\frac{\cos A\cos C+\cos A\cos C}{\cos A\sin C+\cos A\sin C}=\cot C \qquad (\because A+B+C=\pi)$

25. $\frac{\sin\alpha-\sin\gamma}{\cos\gamma-\cos\alpha}=\frac{2\cos\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}{2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}=\cot\left(\frac{\alpha+\gamma}{2}\right)=\cot\beta$

26. $\cos\frac{x}{256}\cdot\cos\frac{x}{128}\cos\frac{x}{64}\cdots\cdots\cos\frac{x}{4}\cdot\cos\frac{x}{2}=\frac{\sin x}{256\sin\left(\frac{x}{256}\right)}$

27. $\frac{(\sin 7\alpha+\sin 5\alpha)+5(\sin 5\alpha+\sin 3\alpha)+12(\sin 3\alpha+\sin\alpha)}{\sin 6\alpha+5\sin 4\alpha+12\sin 2\alpha}$
$$=\frac{2\sin 6\alpha\cos\alpha+5(2\sin 4\alpha\cos\alpha)+12(2\sin 2\alpha\cos\alpha)}{\sin 6\alpha+5\sin 4\alpha+12\sin 2\alpha}=2\cos\alpha$$

28. $\tan^2 A+\tan^2 B+\tan^2 C=\tan A\tan B+\tan B\tan C+\tan A\tan C$

$\Rightarrow \quad \tan A=\tan B=\tan C$

$\Rightarrow \quad A=B=C=\frac{\pi}{3}$

29. $\log_{|\sin x|}|\cos x|+\log_{|\cos x|}|\sin x|=2 \quad \Rightarrow \quad \log_{|\sin x|}|\cos x|=1 \Rightarrow |\cos x|=|\sin x|$

30. $f(x)=\sin^6 x+\cos^6 x=1-3\sin^2 x\cos^2 x=1-\frac{3}{4}\sin^2 2x$

31. $y=\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}\times\left[\dfrac{(1+\sin\alpha)-\cos\alpha}{(1+\sin\alpha)-\cos\alpha}\right]=\dfrac{2\sin\alpha[(1+\sin\alpha)-\cos\alpha]}{(1+\sin\alpha)^2-\cos^2\alpha}$

$=\dfrac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha}$

32. $\dfrac{\tan^3 A}{1+\tan^2 A}+\dfrac{\cot^3 A}{1+\cot^2 A}=\dfrac{\sin^3 A}{\cos A}+\dfrac{\cos^3 A}{\sin A}$

$=\dfrac{\sin^4 A+\cos^4 A}{\sin A\cos A}=\dfrac{1-2\sin^2 A\cos^2 A}{\sin A\cos A}$

$=\sec A\operatorname{cosec} A-2\sin A\cos A$

33. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}=\dfrac{2}{\sqrt{1-\sin^2\theta}}=\dfrac{2}{|\cos\theta|}$

34. $y=(\sin^2\theta+\operatorname{cosec}^2\theta+2)+(\cos^2\theta+\sec^2\theta+2)=7+(\tan^2\theta+\cot^2\theta)\geq 9$

35. $\log_3\sin x-\log_3\cos x-\log_3(1-\tan x)-\log_3(1+\tan x)=-1$

$\log_3\left(\dfrac{\tan x}{1-\tan^2 x}\right)=-1 \Rightarrow \dfrac{\tan x}{1-\tan^2 x}=\dfrac{1}{3} \Rightarrow \tan 2x=\dfrac{2}{3}$

36. $\sin\theta+\operatorname{cosec}\theta=2 \Rightarrow \sin\theta=\operatorname{cosec}\theta=1$; $\left(x+\dfrac{1}{x}\geq 2\right)$

37. $(\tan\theta+\cot\theta)(\tan^2\theta+\cot^2\theta-1)=52$

$(\tan\theta+\cot\theta)\{(\tan\theta+\cot\theta)^2-3\}=52$

Let $\tan\theta+\cot\theta=t$

$t^3-3t-52=0 \Rightarrow t=4$

$\tan^2\theta+\cot^2\theta=(\tan\theta+\cot\theta)^2-2=14$

38. $-5\leq 3\sin x-4\cos x\leq 5$

$10\leq 3\sin x-4\cos x+15\leq 20$

$\log_{20}10\leq\log_{20}(3\sin x-4\cos x+15)\leq\log_{20}20$

39. $x^2+y^2=9$

Let $x=3\cos\theta,\ y=3\sin\theta$

$4a^2+9b^2=16$

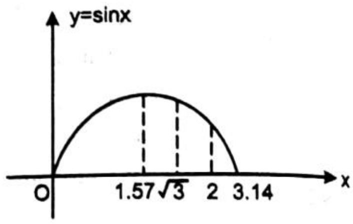
Let $a=2\cos\phi,\ b=\dfrac{4}{3}\sin\phi$

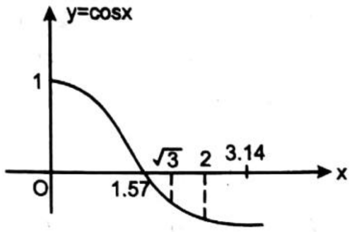
$4a^2x^2+9b^2y^2-12abxy=(2ax-3by)^2$

$=(12\cos\theta\cos\phi-12\sin\theta\sin\phi)^2=144\cos^2(\theta+\phi)$

40. $A^2 = \sin 2 - \sin\sqrt{3}$

$$\sin\sqrt{3} > \sin 2$$
$$A^2 < 0$$
$$B^2 = \cos 2 - \cos\sqrt{3}$$
$$\cos\sqrt{3} > \cos 2$$
$$B^2 < 0$$





Both A and B are not real numbers.

41. $(2^x + 2^{-x} - 2\cos x)(3^{x+\pi} + 3^{-x-\pi} + 2\cos x)(5^{\pi-x} + 5^{x-\pi} - 2\cos x) = 0$

If $\dfrac{2^x + 2^{-x}}{2} = \cos x \quad \Rightarrow \quad x = 0$

If $\dfrac{3^{x+\pi} + 3^{-x-\pi}}{2} = -\cos x \quad \Rightarrow \quad x = -\pi$

If $\dfrac{5^{\pi-x} + 5^{x-\pi}}{2} = \cos x \quad$ (Not possible)

There are two real values of x.

42. $e^{\sin x} - e^{-\sin x} - 4 = 0$

$$e^{2\sin x} - 4e^{\sin x} - 1 = 0$$

$\Rightarrow \quad e^{\sin x} = 2 \pm \sqrt{5}$

If $\quad e^{\sin x} = 2 + \sqrt{5}$

$\Rightarrow \quad \sin x = \ln(2 + \sqrt{5}) \quad$ [$\ln(2+\sqrt{5}) > 1$, Not possible]

If $\quad e^{\sin x} = 2 - \sqrt{5} \quad$ $(2 - \sqrt{5} < 0)$ Not possible

There is no solution.

43. $\sqrt{4\sin^4\alpha + 4\sin^2\alpha \cdot \cos^2\alpha} + 4\cos^2(\pi/4 - \alpha/2)$

$$= \sqrt{4\sin^2\alpha} + 2[1 + \cos(\pi/2 - \alpha)]$$
$$= 2|\sin\alpha| + 2 + 2\sin\alpha$$
$$= -2\sin\alpha + 2 + 2\sin\alpha = 2 \qquad \left(\text{If } \pi < \alpha < \frac{3\pi}{2} \text{ then } \sin\alpha < 0\right)$$

44. $\left(\cos\frac{\pi}{12}-\sin\frac{\pi}{12}\right)\left(\frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}}+\frac{\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}\right)$

$$=\frac{\cos\frac{\pi}{12}-\sin\frac{\pi}{12}}{\sin\frac{\pi}{12}\cdot\cos\frac{\pi}{12}}=\frac{2\sqrt{1-\sin\pi/6}}{\sin\pi/6}=2\sqrt{2}$$

45. $\tan(100°+125°)=\frac{\tan 100°+\tan 125°}{1-\tan 100°\tan 125°}=1$

$\Rightarrow \quad \tan 100°+\tan 125°+\tan 100°\tan 125°=1$

46. If $\sin x+\sin^2 x=1 \Rightarrow \sin x=1-\sin^2 x \Rightarrow \sin x=\cos^2 x$

$$\cos^8 x+2\cos^6 x+\cos^4 x=\sin^4 x+2\sin^3 x+\sin^2 x$$
$$=\sin^2 x(\sin^2 x+2\sin x+1)$$
$$=(1-\sin x)(2+\sin x)$$
$$=2-\sin x-\sin^2 x=1$$

47. Let $x=5\cos\theta,\ y=5\sin\theta$

$$0<3x+4y\le 25 \qquad (\because 3x+4y>0)$$

48. $5\cos 2\theta+2\cos^2\left(\frac{\theta}{2}\right)+1=0$

$$10\cos^2\theta+\cos\theta-3=0 \quad\Rightarrow\quad \cos\theta=\frac{1}{2},\ -\frac{3}{5}$$

49. $\sin\beta=\frac{4}{5}$ where $0<\beta<\pi$ and $\tan\beta>0$

then $\quad \cos\beta=\frac{3}{5}$

$$5\left[\frac{3}{5}\sin(\alpha+\beta)-\frac{4}{5}\cos(\alpha+\beta)\right]\operatorname{cosec}\alpha=5$$

50. $\sin\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)=\sqrt{2}\left[\cos\frac{\pi}{4}\sin\left(x-\frac{\pi}{6}\right)+\sin\frac{\pi}{4}\cdot\cos\left(x+\frac{\pi}{6}\right)\right]=\sqrt{2}\sin\left(x+\frac{5\pi}{12}\right)$

This attained maximum value when $x+\frac{5\pi}{12}=\frac{\pi}{2} \Rightarrow x=\frac{\pi}{12}$

51.
$$\sin 2x-\cos 2x=2a-1$$
$$-\sqrt{2}\le 2a-1\le\sqrt{2}$$
$$\frac{1-\sqrt{2}}{2}\le a\le\frac{1+\sqrt{2}}{2}$$

52.
$$(\cos 12°\cdot\cos 24°\cdot\cos 48°\cdot\cos 84°)(\cos 36°\cos 72°)\cdot\cos 60°$$
$$(-\cos 12°\cdot\cos 24°\cdot\cos 48°\cdot\cos 96°)(\cos 36°\cos 72°)\cdot\cos 60°$$

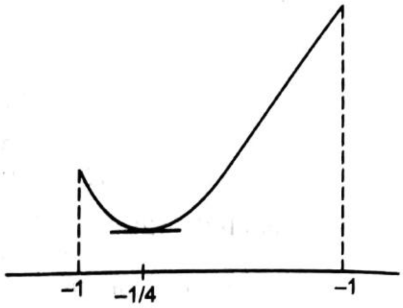
$$\left[-\frac{\sin(2^4\times 12^\circ)}{2^4\sin 12^\circ}\right]\times\left(\frac{\sqrt{5}+1}{4}\times\frac{\sqrt{5}-1}{4}\right)\times\frac{1}{2}=\frac{1}{128}$$

53. $2\cos^2\theta+\cos\theta+1$

$y_{\min}=\frac{7}{8}$ at $\cos\theta=-\frac{1}{4}$

$y_{\max}=4$ at $\cos\theta=1$

$$\frac{y_{\max}}{y_{\min}}=\frac{32}{7}$$

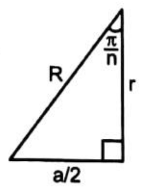


54. $\tan x\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)+1<0;\ \frac{\tan^4 x-1}{3\tan^2 x-1}<0$

$$\Rightarrow \frac{(\tan^2 x+1)(\tan x+1)(\tan x-1)}{(\sqrt{3}\tan x+1)(\sqrt{3}\tan x-1)}<0$$

$$\Rightarrow \frac{\pi}{6}<x<\frac{\pi}{4}$$

55. $a=2R\sin\frac{\pi}{n}=2r\tan\frac{\pi}{n}$



56. $(\cos 12^\circ+\cos 132^\circ)+(\cos 84^\circ+\cos 156^\circ)$

$$=2\cos\frac{12^\circ+132^\circ}{2}\cos\frac{12^\circ-132^\circ}{2}+2\cos\frac{84^\circ+156^\circ}{2}\cos\frac{84^\circ-156^\circ}{2}$$

$$=2\cos 72^\circ\cos 60^\circ+2\cos 120^\circ\cos 36^\circ$$

$$=2\times\frac{\sqrt{5}-1}{4}\times\frac{1}{2}+2\times\left(-\frac{1}{2}\right)\times\frac{\sqrt{5}+1}{4}=-\frac{1}{2}$$

57. $\frac{1}{2}\left[\frac{2\sin\theta\cos\theta}{\cos\theta\cos 3\theta}+\frac{2\sin 3\theta\cos 3\theta}{\cos 9\theta\cos 3\theta}+\frac{2\sin 9\theta\cos 9\theta}{\cos 9\theta\cos 27\theta}+\frac{2\sin 27\theta\cos 27\theta}{\cos 27\theta\cos 81\theta}\right]$

$$=\frac{1}{2}\left[\frac{\sin(3\theta-\theta)}{\cos\theta\cos 3\theta}+\frac{\sin(9\theta-3\theta)}{\cos 3\theta\cos 9\theta}+\frac{\sin(27\theta-9\theta)}{\cos 9\theta\cos 27\theta}+\frac{\sin(81\theta-27\theta)}{\cos 27\theta\cos 81\theta}\right]$$

$$=\frac{1}{2}[\tan 81\theta-\tan\theta]=\frac{1}{2}\left[\frac{\sin 80\theta}{\cos\theta\cos 81\theta}\right]$$

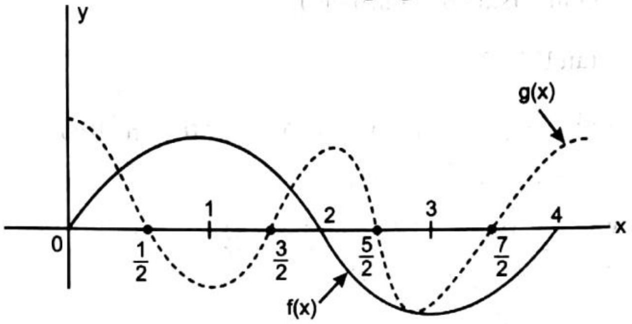
58. $\sin 20°\left(\frac{4\cos 20° + 1}{\cos 20°}\right) = \frac{2\sin 40° + \sin 20°}{\cos 20°} = \frac{2\sin(60° - 20°) + \sin 20°}{\cos 20°} = \sqrt{3}$

59. Let us draw the graph of

$$f(x) = \sin\left(\frac{x\pi}{2}\right)$$

and $$g(x) = \cos(x\pi)$$

On the same xy-plane as shown in the following figure.



From this graphical representation, it is clear that y is strictly increasing in $\left(\frac{5}{2}, \frac{7}{2}\right)$

Because for all values of x,

$$\frac{5}{2} < x < \frac{7}{2}$$

That is, $$\sin\left(\frac{x\pi}{2}\right) < 0$$

and $$\cos(x\pi) < 0$$

which imply that $$\frac{dy}{dx} > 0$$

which means that y is strictly increasing.

60. $8\sin\theta\sin 3\theta\left(\frac{\sin 8\theta}{4\sin 2\theta}\right) = \cos 6\theta$

$$\sin 3\theta \sin 8\theta = \cos 6\theta \cos\theta$$
$$\cos 5\theta - \cos 11\theta = \cos 7\theta + \cos 5\theta$$
$$\cos 7\theta + \cos 11\theta = 0$$
$$2\cos 9\theta \cdot \cos 2\theta = 0$$

61. $\tan A = -\frac{1}{3} \Rightarrow \sin A = \frac{1}{\sqrt{10}}$; $\cos A = -\frac{3}{\sqrt{10}}$

63. $(2\cos\theta)^2 = (1 - \sin\theta)^2 \Rightarrow \sin\theta = 1$ or $\sin\theta = \frac{-3}{5}$

64. $\sin\theta + \dfrac{1}{\sin\theta} = 2 \Rightarrow \sin\theta = 1$

65. $\tan^2\theta + \cot^2\theta = a \Rightarrow \tan^3\theta + \cot^3\theta = \sqrt{a+2}\,(a-1) = 52$

66. $\tan A = -\tan C = \dfrac{5}{12}$

$\cos B = -\cos D = -\dfrac{3}{5} \quad \Rightarrow \tan D = \dfrac{4}{3}$

67. $\sqrt{\tan^2\theta - \sin^2\theta} = \sqrt{\tan^2\theta\sin^2\theta} = |\tan\theta\sin\theta|$

68. $\dfrac{\sin 10° + \sin 20°}{\cos 10° + \cos 20°} = \tan 15° = 2 - \sqrt{3}$

69. $(\sin^2\theta)^3 + (\cos^2\theta)^3 = (\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)$

$$= 1 - 3\sin^2\theta\cos^2\theta$$

70. $\dfrac{\tan x + 1}{\tan x - 1} - \dfrac{\sec^2 x + 2}{\tan^2 x - 1} \Rightarrow \dfrac{(\tan x + 1)^2 - (\sec^2 x + 2)}{\tan^2 x - 1}$

$$\Rightarrow \frac{2\tan x - 2}{\tan^2 x - 1} \Rightarrow \frac{2}{\tan x + 1}$$

71. $\dfrac{\cot\alpha - \tan\alpha}{\cot\alpha + \tan\alpha} - [\cos 450° + \cos(2\alpha - 180°)]$

$\Rightarrow \quad (\cos^2\alpha - \sin^2\alpha) + \cos 2\alpha = 2\cos 2\alpha$

72. $\left(\dfrac{1+\tan\alpha}{1-\tan\alpha}\right)\cdot\left(\dfrac{1+\tan\alpha}{1-\tan\alpha}\right) + 1$

$1 + \tan^2\left(\dfrac{\pi}{4} + \alpha\right) = \sec^2\left(\dfrac{\pi}{4} + \alpha\right) = \text{cosec}^2\left(\dfrac{\pi}{4} - \alpha\right)$

73. $\dfrac{\tan\alpha + \sin\alpha}{1 + \cos\alpha} = \tan\alpha$

74. $(\cos 2\alpha + \cos 5\alpha) - (\cos 3\alpha + \cos 4\alpha)$

$2\cos\dfrac{7\alpha}{2}\cdot\cos\dfrac{3\alpha}{2} - 2\cos\dfrac{7\alpha}{2}\cdot\cos\dfrac{\alpha}{2}$

$2\cos\dfrac{7\alpha}{2}\left[\cos\dfrac{3\alpha}{2} - \cos\dfrac{\alpha}{2}\right] = -4\sin\dfrac{\alpha}{2}\sin\alpha\cos\dfrac{7\alpha}{2}$

75. $\cos 2\gamma = \dfrac{1 - \tan^2\gamma}{1 + \tan^2\gamma} = \dfrac{1 - \left(\dfrac{1 + \sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)^2}{1 + \left(\dfrac{1 + \sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)^2}$

$\Rightarrow \quad \dfrac{(\cos\alpha\cos\beta)^2 - (1 + \sin\alpha\cos\beta)^2}{(\cos\alpha\cos\beta)^2 + (1 + \sin\alpha\sin\beta)^2} = \dfrac{[1 + \cos(\alpha - \beta)][\cos(\alpha + \beta) - 1]}{(\cos\alpha\cos\beta)^2 + (1 + \sin\alpha\sin\beta)^2} \le 0$

76. $x=\frac{2\pi}{3}$ (IInd quadrant)

$$\cos x+\cos 2x+\cos 3x+\ldots+\cos 100x=\frac{\sin 50x}{\sin\frac{x}{2}}\cdot\cos\left(\frac{101x}{2}\right)=-\frac{1}{2}$$

77. $\cos^3 0°+\cos^3\frac{\pi}{3}+\cos^3\frac{2\pi}{3}+\cos^3\pi+\ldots+\cos^3\frac{10\pi}{3}=-\frac{1}{8}$

78. $\frac{1-2(\cos 60°-\cos 80°)}{2\sin 10°}=\frac{2\cos 80°}{2\sin 10°}=1$

79. $(x+5)^2+(y-12)^2=14^2$

Let $x=-5+14\cos\theta,\ y=12+14\sin\theta$

$\Rightarrow\ x^2+y^2=365+336\sin\theta-140\cos\theta$

80. $\tan\theta=\lambda$ has three distinct solution in $[0,2\pi]\Rightarrow\lambda=0$ and $\theta=0,\pi,2\pi$.

81. $\sqrt{\frac{1+\tan\alpha}{1-\tan\alpha}}+\sqrt{\frac{1-\tan\alpha}{1+\tan\alpha}}=\frac{2}{\sqrt{1-\tan^2\alpha}}$

82. $3\sin\theta+4\cos\theta=5\left(\frac{3}{5}\sin\theta+\frac{4}{5}\cos\theta\right)=5\sin(\theta+53°)$

83. $f(n)=\prod_{r=1}^{n}\cos r$

$f(4)=\cos 1\cdot\cos 2\cdot\cos 3\cdot\cos 4<0$

$f(5)=\cos 1\cdot\cos 2\cdot\cos 3\cdot\cos 4\cdot\cos 5<0$

84. $\frac{(p^2-q^2)^2}{pq}=\frac{(4\tan A\sin A)^2}{\tan^2 A-\sin^2 A}=16$

85. $0<\sin\alpha<\cos\alpha<1\qquad \alpha\in\left(0,\frac{\pi}{4}\right)$

$(\sin\alpha)^{\cos\alpha}<(\sin\alpha)^{\sin\alpha}$

$(\cos\alpha)^{\cos\alpha}<(\cos\alpha)^{\sin\alpha}$

86. $32\sin\frac{A}{2}\sin\frac{5A}{2}=16(\cos 2A-\cos 3A)$

$$=16[(2\cos^2 A-1)-(4\cos^3 A-3\cos A)]$$

87. $\cos\alpha\cos\beta-\sin\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta=0$

$\cos\alpha(\cos\beta-\sin\beta)+\sin\alpha(\cos\beta-\sin\beta)=0$

$(\cos\beta-\sin\beta)(\cos\alpha+\sin\alpha)=0$

$\cos\alpha=-\sin\alpha\qquad(\because\ \cos\beta\neq\sin\beta)$

$\tan\alpha=-1$

88. $2^x = 3^y = 6^{-z} = k$

$x = \log_2 k, y = \log_3 k, z = -\log_6 k$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

89. $(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = \left(\frac{-21}{65}\right)^2 + \left(\frac{-27}{65}\right)^2$

$2 + 2\cos(\alpha - \beta) = \frac{1170}{(65)^2} = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

90. $\mu^2 = a^2 + b^2 + 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\theta + b^2\cos^2\theta)}$

$= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^4 + b^4 - 2a^2b^2)\sin^2\theta\cos^2\theta}$

91. $Q = \sum_{r=0}^{n} \frac{\sin(3^r\theta)\cos(3^r\theta)}{\cos(3^r\theta)\cos(3^{r+1}\theta)} = \frac{1}{2}\sum_{r=0}^{n} \tan(3^{r+1}\theta) - \tan(3^r\theta) = \frac{1}{2}P$

92. When $270° < \theta < 360°$, we have

$$\sqrt{2(1+\cos\theta)} = \sqrt{\left(2\cos^2\frac{\theta}{2}\right)}$$

which is non-negative. Now, the above equation can be written as

$$\sqrt{2(1+\cos\theta)} = 2\left|\cos\frac{\theta}{2}\right|$$

$$= -2\cos\frac{\theta}{2} \qquad \left(\because \cos\frac{\theta}{2} < 0 \text{ when } 135° < \frac{\theta}{2} < 180°\right)$$

Now, let us consider that $\sqrt{2 + \sqrt{2(1+\cos\theta)}}$

which is not-negative. That is,

$$\sqrt{2 + \sqrt{2(1+\cos\theta)}} = \sqrt{2 - 2\cos\frac{\theta}{2}}$$

$$= \sqrt{2}\sqrt{1 - \cos\frac{\theta}{2}} = \sqrt{2}\sqrt{2\sin^2\frac{\theta}{4}}$$

$$= 2\left|\sin\frac{\theta}{4}\right|$$

$$= 2\sin\frac{\theta}{4} \qquad \left(\because \sin\frac{\theta}{4} > 0 \text{ when } \frac{135°}{2} < \frac{\theta}{4} < 90°\right)$$

93. We know that $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$

When $x = -\frac{3\pi}{4}$, we have $\sin x + \cos x = -\sqrt{2}$

when $x = -\frac{3\pi}{4}$, we have $y = -\sqrt{2} + 1 < 0$

which implies that options (1) and (2) are incorrect.

Now, at $x = \frac{\pi}{4}$, we have $\sin x + \cos x = \sqrt{2}$

That is, $(\sin 4x + \cos 4x)^2 \neq 2$. Therefore, $y \neq \sqrt{2} + 2$ for any $x \in R$.

which implies that option (4) is incorrect.

Note : The maximum value of $\sin x + \cos x$ is $\sqrt{2}$, for $x = \frac{\pi}{4}$ and the maximum value of $(\sin 4x + \cos 4x)^2$ is 2, for $x = \frac{\pi}{16}$.

94. $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = (-\cos z)^2 + (-\sin z)^2$

$$2 + 2\cos(x - y) = 1$$

95. $$\frac{1}{\sin 10°} + \frac{1}{\sin 50°} - \frac{1}{\sin 70°} = \frac{\sin 50° \sin 70° + \sin 10° \sin 70° - \sin 10° \sin 50°}{\sin 10° \sin 50° \sin 70°}$$

$$= \frac{\frac{1}{2}(\cos 20° - \cos 120° + \cos 60° - \cos 80° - \cos 40° + \cos 60°)}{\frac{1}{4}\sin 30°}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{2} + \cos 20° - 2\cos 60° \cos 20°\right)}{\frac{1}{4}\sin 30°} = 6$$

Exercise-2 : One or More than One Answer is/are Correct

1. $\cot 12° \cot 24° \cot 48° [\cot 28° \cot(60°-28°) \cot(60°+28°)] = (\cot 12° \cot 48°)(\cot 24° \cot 84°)$

$$= \frac{\cot 36°}{\cot 72°} \times \frac{\cot 72°}{\cot 36°} = 1$$

2. $$\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$$

$\Rightarrow$ $$\cot^4 x - 2\cot^2 x + a^2 - 2 = 0$$

$\Rightarrow$ $$(\cot^2 x - 1)^2 = 3 - a^2$$

to have atleast one solution

$$3 - a^2 \geq 0$$

$\Rightarrow$ $$a^2 - 3 \leq 0$$

$$a \in [-\sqrt{3}, \sqrt{3}]$$

Integral values −1, 0, 1

$\therefore$ $$\text{Sum} = 0$$

3. (A) $$\tan 1 > \tan^{-1} 1 \Rightarrow \tan 1 > \frac{\pi}{4}$$

(B) $$\sin 1 > \cos 1$$
$$\sin 57.3° > \cos 57.3°$$

(C) $$\tan 1 < \sin 1 \quad \text{(not possible)}$$

Because $\tan 57.3 > 1 > \sin 57.3°$

(D) $$\cos 1 < \frac{\pi}{4}$$

$\Rightarrow$ $$\cos(\cos 1) > \cos\left(\frac{\pi}{4}\right)$$

4. (A) $\tan 1 > 1$ and $\sin 1 < 1$, then $\log_{\sin 1} \tan 1 < 0$

(B) $1 + \tan 3 < 1$ and $\cos 1 < 1$, then $\log_{\cos 1}(1 + \tan 3) > 0$

(C) $\cos\theta + \sec\theta > 2$ and $\log_{10} 5 < 1$, then $\log_{\log 10} 5(\cos\theta + \sec\theta) < 0$

(D) $2\sin 18° < 1$ and $\tan 15° < 1$, then $\log_{\tan 15°} 2\sin 18° > 0$

5. Put $\sin\alpha = \dfrac{2\tan\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)}$, $\cos\alpha = \dfrac{1 - \tan^2\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)}$

6. Given $\dfrac{\sin(2\alpha + \beta)}{\sin\beta} = \dfrac{3}{1}$

Option (C) $\dfrac{\sin(2\alpha + \beta) + \sin\beta}{\sin(2\alpha + \beta) - \sin\beta} = \dfrac{3 + 1}{3 - 1}$ (Use C and D method)

$$\tan(\alpha + \beta) = 2\tan\alpha$$

Option (B) $$3\sin\beta = \sin(2\alpha + \beta)$$
$$2\sin\beta = \sin(2\alpha + \beta) - \sin\beta$$
$$2\sin\beta = 2\cos(\alpha + \beta)\sin\alpha$$

Option (D) $$3\sin\beta = \sin\{\alpha + (\alpha + \beta)\}$$
$$3\sin\beta = \sin\alpha\cos(\alpha + \beta) + \cos\alpha\sin(\alpha + \beta)$$

Subtract from (B) option
$$2\sin\beta = \cos\alpha\sin(\alpha + \beta)$$

Option (A) $\cot\beta - 3\cot(2\alpha + \beta) = \dfrac{\cos\beta}{\sin\beta} - 3\dfrac{\cos(2\alpha + \beta)}{\sin(2\alpha + \beta)}$

$$= \frac{\cos\beta}{\sin\beta} - 3\frac{\cos(2\alpha + \beta)}{3\sin\beta} = \frac{2\sin(\alpha + \beta)\sin\alpha}{\sin\beta} = 4\tan\alpha \quad \text{(from D)}$$

Also $$\cot\alpha + \cot(\alpha + \beta) = \frac{3}{2}\cot\alpha \quad \text{(from C)}$$

Now multiply the two relations.

7. $$\sin(x+20^\circ)=\sin(x+40^\circ)+\sin(x-40^\circ)$$

$$\sin(x+20^\circ)-\sin(x-40^\circ)=\sin(x+40^\circ)$$

$$\cos(x-10^\circ)=\sin(x+40^\circ)=\cos[90^\circ-(x+40^\circ)]$$

$\Rightarrow$ $x=30^\circ$ now check the option, only (a) and (b) satisfy

8. $$2\Sigma(\cos x\cos y)+2\Sigma(\sin x\sin y)+3=0$$

$$(\Sigma\cos x)^2+(\Sigma\sin x)^2=0$$

$\Rightarrow$ $\Sigma\cos x=0$ and $\Sigma\sin x=0$

$$\cos 3x+\cos 3y+\cos 3z=4(\cos^3 x+\cos^3 y+\cos^3 z)-3(\cos x+\cos y+\cos z)$$

$$=12\cos x\cos y\cos z$$

9. $0<\sin x<1,\ \ 0<\cos x<1$

If $\sin^n x+\cos^n x=1$ $\quad n=2$

$\sin^n x+\cos^n x>1$ $\quad n<2$

$\sin^n x+\cos^n x<1$ $\quad n>2$

10. If $$x=\sin(\alpha-\beta)\sin(\gamma-\delta)$$

$$2x=\cos(\alpha-\beta-\gamma+\delta)-\cos(\alpha-\beta+\gamma-\delta)$$

$$y=\sin(\beta-\gamma)\sin(\alpha-\delta)$$

$\Rightarrow$ $$2y=\cos(\beta-\gamma-\alpha+\delta)-\cos(\beta-\gamma+\alpha-\delta)$$

$$z=\sin(\gamma-\alpha)\sin(\beta-\delta)$$

$\Rightarrow$ $$2z=\cos(\gamma-\alpha-\beta+\delta)-\cos(\gamma-\alpha+\beta+\delta)$$

$$2x+2y+2z=0 \quad\Rightarrow\quad x+y+z=0$$

If $x+y+z=0$ then $x^3+y^3+z^3=3xyz$

11. $$X^2+4XY+Y^2=(x\cos\theta-y\sin\theta)^2+(x\sin\theta+y\cos\theta)^2$$

$$+4(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)$$

$$=x^2+y^2+4\{x^2\sin\theta\cos\theta-y^2\sin\theta\cos\theta+xy(\cos^2\theta-\sin^2\theta)\}$$

$$=x^2(1+4\sin\theta\cos\theta)+y^2(1-4\sin\theta\cos\theta)+4xy(\cos^2\theta-\sin^2\theta)$$

$$\cos^2\theta-\sin^2\theta=0$$

$\Rightarrow$ $$\theta=\frac{\pi}{4} \qquad (0\le\theta\le\pi/2)$$

$$x^2+4XY+Y^2=3x^2-y^2$$

$\Rightarrow$ $A=3$ and $B=-1$

12. (A) $$2(a+d)=2(b+c)$$

(B) $$\tan 50^\circ=\frac{1}{\tan 40^\circ}=\frac{1-\tan^2 20^\circ}{2\tan 20^\circ}$$

$\Rightarrow$ $$\tan 20^\circ+2\tan 50^\circ=\tan 70^\circ$$

$\Rightarrow \qquad 2a + 2b = 2c$

(D) $\tan 20° - 2\tan 10° = \tan 20° \tan^2 10° > 0$

$\Rightarrow \quad \tan 20° > 2\tan 10°$

$\Rightarrow \quad b > a$ and $d > c$

13. (A) $\frac{1}{2}(2\sin 75° \cos 75°) = \frac{1}{2}\sin 150° = \frac{1}{4}$

(B) $\log_2^{28} = 2 + \log_2^7$ (irrational)

(C) $\log_3^5 \cdot \log_5^6 = \log_3^6 = 1 + \log_3^2$ (irrational)

(D) $8^{-\log_{27}^3} = 8^{-1/3} = \frac{1}{2}$

14. $\alpha - \beta = \sin x \cos x(\cos^2 x - \sin^2 x) = \frac{1}{2}\sin 2x \cos 2x = \frac{1}{4}\sin 4x$

$\alpha + \beta = \sin x \cos x = \frac{1}{2}\sin 2x$

15.
$$\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}}$$
$$= \sqrt{2 + \sqrt{4\cos^2 2\theta}}$$
$$= \sqrt{2 + 2|\cos 2\theta|}$$

If $\pi < 2\theta < 3\pi/2$ then $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$

$$\sqrt{2 + 2|\cos 2\theta|} = \sqrt{2 - 2\cos 2\theta} = 2|\sin\theta| = 2\sin\theta$$

If $\frac{3\pi}{2} < 2\theta < 2\pi$ then $\frac{3\pi}{4} < \theta < \pi$

$$\sqrt{2 + 2|\cos 2\theta|} = \sqrt{2 + 2\cos 2\theta} = 2|\cos\theta| = -2\cos\theta$$

16. $1 + \tan\alpha + \tan^2\alpha = \tan^3\alpha$

$\Rightarrow \quad 1 + \tan^2\alpha = \tan\alpha(\tan^2\alpha - 1)$

18. $\alpha > \frac{1}{\sin^6 x + \cos^6 x} \Rightarrow \alpha > \frac{1}{1 - 3\sin^2 x \cos^2 x}$; $1 \le \frac{1}{1 - 3\sin^2 x \cos^2 x} \le 4$

19. $\log_{10}\sin x + \log_{10}\cos x + 2\log_{10}\cot x + \log_{10}\tan x = -1$

$\log_{10}(\sin x \cdot \cos x \cdot \cot x) = k = \log_{10}\cos^2 x = -1$

20. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\frac{3}{\tan C} + \frac{6}{\tan C} + \tan C = \frac{3}{\tan C} \cdot \frac{6}{\tan C} \cdot \tan C$

$\Rightarrow \qquad \tan^2 C = 9 \Rightarrow \tan C = 3$

21. $\frac{(1-\cot x)}{\sin^2 x}=(1-\cot x)\cdot \text{cosec}^2 x$

$=(1-\cot x)(1+\cot^2 x)$

22. $f(x)=\frac{1}{2}\left[2\sin^2 x+2\sin^2\left(x+\frac{2\pi}{3}\right)+2\sin^2\left(x+\frac{4\pi}{3}\right)\right]$

23. $y=\frac{\tan x}{\tan 3x}=\frac{1-3\tan^2 x}{3-\tan^2 x}$

$\tan^2 x=\frac{1-3y}{3-y}>0$

24. $\sqrt{2}\sin(A-B)=\cos B(\sin B-\sin^3 B)-\sin B(\cos B+\cos^3 B)$

$=-\sin B\cos B$

$=-\frac{1}{2}\sin 2B \Rightarrow \sin(A-B)=-\frac{\sin 2B}{2\sqrt{2}}$

25. $\alpha>\frac{1}{\sin^6 x+\cos^6 x}\Rightarrow \alpha>\frac{1}{1-3\sin^2 x\cos^2 x}$; $1\le \frac{1}{1-3\sin^2 x\cos^2 x}\le 4$

26. $1+\tan\alpha+\tan^2\alpha=\tan^3\alpha$

$\Rightarrow\ 1+\tan^2\alpha=\tan\alpha(\tan^2\alpha-1)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

1. $\theta=286.5°$ (IV quadrant) $l<0, m>0$

2. $\tan(-1042°)=-\tan(1080°-38°)=\tan 38°<\tan 45°$

3. $\theta=401.1°$ (I quadrant) $l>0, m>0$

Paragraph for Question Nos. 4 to 6

$a=\sin\alpha \quad b=\sin\left(\alpha+\frac{2\pi}{3}\right) \quad c=\sin\left(\alpha+\frac{4\pi}{3}\right)$

$p=\cos\alpha \quad q=\cos\left(\alpha+\frac{2\pi}{3}\right) \quad r=\cos\left(\alpha+\frac{4\pi}{3}\right)$

4. $a+b+c=\sin\alpha+\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)$

$$=\sin\alpha+2\sin(\alpha+\pi)\cos\left(\frac{\pi}{3}\right)=0$$

5. $ab+bc+ac=\sin\alpha\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{2\pi}{3}\right)\sin\left(\alpha+\frac{4\pi}{3}\right)+\sin\alpha\sin\left(\alpha+\frac{4\pi}{3}\right)$

$$=\frac{1}{2}\left[\cos\frac{2\pi}{3}-\cos\left(2\alpha+\frac{2\pi}{3}\right)+\cos\frac{2\pi}{3}-\cos(2\alpha+2\pi)+\cos\frac{4\pi}{3}-\cos\left(2\alpha+\frac{4\pi}{3}\right)\right]=\frac{-3}{4}$$

6. $qc-rb=\cos\left(\alpha+\frac{2\pi}{3}\right)\sin\left(\alpha+\frac{4\pi}{3}\right)-\cos\left(\alpha+\frac{4\pi}{3}\right)\sin\left(\alpha+\frac{2\pi}{3}\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

Paragraph for Question Nos. 7 to 8

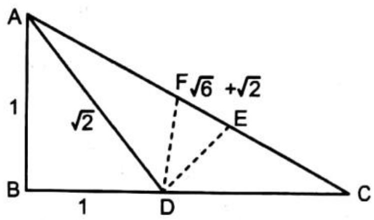
7.

$$\tan A=\sqrt{7+4\sqrt{3}}=\cot C$$

$$\sqrt{\tan A+\cot C}=\sqrt{2\sqrt{7+4\sqrt{3}}}$$

$$=\sqrt{2(2+\sqrt{3})}=\sqrt{4+2\sqrt{3}}$$

$$=\sqrt{3}+1$$



8. $\log_{AE}\left(\frac{AC}{CD}\right)=\log_{\sqrt{2}}\left(\frac{\sqrt{2}+\sqrt{6}}{1+\sqrt{3}}\right)=\log_{\sqrt{2}}\sqrt{2}=1$

Paragraph for Question Nos. 9 to 10

9. In a ΔABC, $\cot A+\cot B+\cot C\geq\sqrt{3}\Rightarrow\cot\theta\geq\sqrt{3}$

10. $\cot\theta-\cot A=\cot B+\cot C\Rightarrow\sin(A-\theta)=\frac{\sin^2 A\sin\theta}{\sin B\sin C}$

$\sin(B-\theta)=\frac{\sin^2 B\cdot\sin\theta}{\sin A\sin C}$ and $\sin(C-\theta)=\frac{\sin^2 C\sin\theta}{\sin A\sin B}$

Paragraph for Question Nos. 11 to 12

11. $f(x)=\frac{\left|\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}\right|}$

12. If $\frac{\pi}{2}<\frac{x}{2}<\pi\Rightarrow f(x)=\frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}=\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}$

Exercise-4 : Matching Type Problems

1. (A) If $A+B=45°$ then $(1+\tan A)(1+\tan B)=2$

(B) $a^2-5a\le 6\sin x\ \ \forall\ \ x\in R$

$a^2-5a\le -6$

$a^2-5a+6\le 0 \quad\Rightarrow\quad (a-3)(a-2)\le 0$

(C) $\dfrac{\left(a+\frac{1}{a}\right)^4-\left(a^4+\frac{1}{a^4}+2\right)}{\left(a+\frac{1}{a}\right)^2+a^2+\frac{1}{a^2}}=\dfrac{\left(a+\frac{1}{a}\right)^4-\left(a^2+\frac{1}{a^2}\right)^2}{\left(a+\frac{1}{a}\right)^2+a^2+\frac{1}{a^2}}=\left(a+\frac{1}{a}\right)^2-\left(a^2+\frac{1}{a^2}\right)=2$

(D) $\sum_{k=1}^{3}(x-k)^2=(x-1)^2+(x-2)^2+(x-3)^2=0$ No real root

2. (A) $y=\dfrac{1-\tan^2(\pi/4-x)}{1+\tan^2(\pi/4-x)}=\cos(\pi/2-2x)=\sin 2x$

(B) $0\le\log_3\left(\dfrac{5\sin x-12\cos x+26}{13}\right)\le 1$

(C) $y=-2\sin^2 x+\cos x+3=2\cos^2 x+\cos x+1=2\left(\cos x+\dfrac{1}{4}\right)^2+\dfrac{7}{8}$

(D) $y=4\sin^2\theta+4\sin\theta\cos\theta+\cos^2\theta=(2\sin\theta+\cos\theta)^2$

4. (A) $\cos^2 x=\left(\dfrac{1}{5}-\sin x\right)^2$

$\Rightarrow\quad (5\sin x-4)(5\sin x+3)=0$

$\Rightarrow\quad \sin x=\dfrac{4}{5}$ or $-\dfrac{3}{5}$

(B) $\cot\dfrac{\theta}{2}=1+\cot\theta$

$\Rightarrow\quad 2\cos^2\dfrac{\theta}{2}=\cos\theta+\sin\theta$

$\Rightarrow\quad \sin\theta=1 \quad\Rightarrow\quad \theta=-\dfrac{3\pi}{2},\dfrac{\pi}{2}$

(C) $f(x)=-\sin^4 x+8\sin^2 x+2$

$\Rightarrow\quad f(x)\in[2,9]$

(D) $\log_2\dfrac{(2x^2+5x+27)}{(2x-1)^2}\ge 0 \qquad \left(x>\dfrac{1}{2}\right)$

$$\Rightarrow \quad 2x^2 - 9x - 26 \le 0$$

$$\Rightarrow \quad -2 \le x \le \frac{13}{2}$$

5. (A) $f(x) = -2\sin^2 x + \sin x - 6$

$y_{min} = -9$ at $\sin x = -1$

$y_{max} = -\frac{47}{8}$ at $\sin x = \frac{1}{4}$

(B) $f(x) = 2\cos^2 x + 6$

$y_{min} = 6;\ \ y_{max} = 8$

(C) $f(x) = \frac{1}{2}[4\sin 2x - 1 + \cos 2x + 3(1 + \cos 2x)]$

$= \frac{1}{2}[2 + 4\sin 2x + 4\cos 2x]$

$= 1 + 2(\sin 2x + \cos 2x)$

$y_{max} = 1 + 2\sqrt{2};\ \ y_{min} = 1 - 2\sqrt{2}$

(D) $f(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + \sin x\right)$

Exercise-5 : Subjective Type Problems

1. $\frac{\sin 80^\circ \sin 65^\circ \sin 35^\circ}{2\sin 35^\circ \cos 15^\circ + 2\sin 35^\circ \cos 35^\circ} = \frac{\sin 80^\circ \sin 65^\circ}{2(\cos 15^\circ + \cos 35^\circ)} = \frac{\sin 80^\circ \sin 65^\circ}{4\cos 25^\circ \cos 10^\circ} = \frac{1}{4}$

2. If $A + B = 45^\circ$

$$(1 - \cot A)(1 - \cot B) = 2$$

$$\Rightarrow \quad (1 - \cot 23^\circ)(1 - \cot 22^\circ) = 2$$

3.

$$4x^2 - 7x + 1 = 0$$

$$\tan A + \tan B = \frac{7}{4}$$

$$\tan A \cdot \tan B = \frac{1}{4}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A + \tan B} = \frac{7}{3}$$

$$4\sin^2(A + B) - 7\sin(A + B)\cos(A + B) + \cos^2(A + B)$$

$$= \frac{4\tan^2(A + B) - 7\tan(A + B) + 1}{1 + \tan^2(A + B)} = 1$$

4. $\frac{(18-2)\times 180^\circ}{18}+\frac{(n-2)\times 180^\circ}{n}+60^\circ=360^\circ \Rightarrow n=9$

5. $10(1-\cos 2\alpha)^2+15(1+\cos 2\alpha)^2=24$

$\Rightarrow \quad (5\cos 2\alpha+1)^2=0 \quad \Rightarrow \quad \cos 2\alpha=-\frac{1}{5} \;\Rightarrow\; \tan^2\alpha=\frac{3}{2}$

6. $\tan\left(\frac{3\pi}{8}-\frac{\pi}{8}\right)\left(\tan\frac{3\pi}{8}-\tan\frac{\pi}{8}\right)+\tan\left(\frac{5\pi}{8}-\frac{3\pi}{8}\right)\left(\tan\frac{5\pi}{8}-\tan\frac{3\pi}{8}\right)$

$+\tan\left(\frac{7\pi}{8}-\frac{5\pi}{8}\right)\left(\tan\frac{7\pi}{8}-\tan\frac{3\pi}{8}\right)+\tan\left(\frac{9\pi}{8}-\frac{7\pi}{8}\right)\left(\tan\frac{9\pi}{8}-\tan\frac{7\pi}{8}\right)=\tan\frac{9\pi}{8}-\tan\frac{\pi}{8}=0$

7. $\dfrac{\cos\frac{2\pi}{7}+2\cos^2\frac{\pi}{7}}{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}}=\dfrac{4\left(\cos\frac{2\pi}{7}+2\cos^2\frac{\pi}{7}\right)\sin\frac{\pi}{7}}{\sin\frac{4\pi}{7}}=\dfrac{4\left(1+2\cos\frac{2\pi}{7}\right)\sin\frac{\pi}{7}}{\sin\frac{3\pi}{7}}$

$$=\frac{4\left(1+2\cos\frac{2\pi}{7}\right)\sin\frac{\pi}{7}}{\sin\frac{\pi}{7}\left(3-4\sin^2\frac{\pi}{7}\right)}=4$$

8.
$$a^2\sec^2 200^\circ=c^2\tan^2 200^\circ+d^2+2cd\tan 200^\circ$$
$$b^2\sec^2 200^\circ=c^2+d^2\tan^2 200^\circ-2cd\tan 200^\circ$$
$\Rightarrow \quad a^2+b^2=c^2+d^2$

$(a\sec 200^\circ-c\tan 200^\circ)^2=d^2$

$(b\sec 200^\circ+d\tan 200^\circ)^2=c^2$

$\Rightarrow \quad (c^2+d^2)(\sec^2 200^\circ+\tan^2 200^\circ)+(2bd-2ac)\sec 200^\circ\tan 200^\circ=c^2+d^2$

$\Rightarrow \quad (c^2+d^2)(2\tan^2 200^\circ)=(2ac-2bd)\sec 200^\circ\tan 200^\circ$

$\Rightarrow \quad \frac{2(c^2+d^2)}{ac-bd}=\frac{2\sec 200^\circ}{\tan 200^\circ}=\frac{2}{\sin 200^\circ}=\frac{-2}{\sin 20^\circ}$

9. $2\cos\frac{\pi}{17}\cos\frac{9\pi}{17}+\cos\frac{7\pi}{17}+\cos\frac{9\pi}{17}=\cos\frac{10\pi}{17}+\cos\frac{8\pi}{17}+\cos\frac{7\pi}{17}+\cos\frac{9\pi}{17}=0$

10. $\dfrac{\cos\theta-\cos 3\theta+\cos 3\theta-\cos 9\theta+\cos 9\theta-\cos 17\theta}{\sin 3\theta-\sin\theta+\sin 9\theta-\sin 3\theta+\sin 17\theta-\sin 9\theta}=\dfrac{\cos\theta-\cos 17\theta}{\sin 17\theta-\sin\theta}=\tan 9\theta$

11. $8abc=8\sin 10^\circ\sin 50^\circ\sin 70^\circ=1$

$$\frac{a+b}{c}=\frac{\sin 10^\circ+\sin 50^\circ}{\sin 70^\circ}=\frac{2\sin 30^\circ\cos 20^\circ}{\sin 70^\circ}=1$$

$\frac{1}{a}+\frac{1}{b}-\frac{1}{c}=\frac{1}{\sin 10^\circ}+\frac{1}{\sin 50^\circ}-\frac{1}{\sin 70^\circ}=\frac{\sin 50^\circ\sin 70^\circ+\sin 10^\circ\sin 70^\circ-\sin 10^\circ\sin 50^\circ}{\sin 10^\circ\sin 50^\circ\sin 70^\circ}=6$

12. $\frac{1}{4}\left[4\sin^3\theta + 4\sin^3\left(\theta+\frac{2\pi}{3}\right) + 4\sin^3\left(\theta+\frac{4\pi}{3}\right)\right]$

$$=\frac{1}{4}\left[3\sin\theta - \sin 3\theta + 3\sin\left\{\left(\theta+\frac{2\pi}{3}\right) - \sin(3\theta+2\pi) + 3\sin\left(\theta+\frac{4\pi}{3}\right) - \sin(3\theta+4\pi)\right\}\right]$$

$$=\frac{1}{4}\left[3\left\{\sin\theta + \sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta+\frac{4\pi}{3}\right)\right\} - 3\sin 3\theta\right] = -\frac{3}{4}\sin 3\theta$$

13. $\sum_{r=1}^{n}\frac{\sin(2^r - 2^{r-1})}{\cos 2^r \cos 2^{r-1}} = \sum_{r=1}^{n}(\tan 2^r - \tan 2^{r-1}) = \tan 2^n - \tan 1$

14. $x = \sec\theta - \tan\theta,\ y = \operatorname{cosec}\theta + \cot\theta$

$$y - x - xy = \frac{1+\cos\theta}{\sin\theta} - \frac{1-\sin\theta}{\cos\theta} - \frac{(1-\sin\theta)(1+\cos\theta)}{\sin\theta\cdot\cos\theta} = 1$$

15.
$$\cos 18° - \cos 72° = 2\sin 45° \sin 27°$$
$$= \sqrt{2}\sin 27°$$

16. $3(\sin 1 - \cos 1)^4 + 6(\sin 1 + \cos 1)^2 + 4(\sin^6 1 + \cos^6 1)$

$$= 3(1 - 2\sin 1\cos 1)^2 + 6(1 + 2\sin 1\cos 1) + 4(1 - 3\sin^2 1\cos^2 1)$$
$$= 3(1 + 4\sin^2 1\cos^2 1 - 4\sin 1\cos 1) + 10 + 12\sin 1\cos 1 - 12\sin^2 1\cos^2 1$$
$$= 13$$

17. $3^{\sin 2x + 2\cos^2 x} + \frac{3^3}{3^{\sin 2x + 2\cos^2 x}} = 28$

Let $3^{\sin 2x + 2\cos^2 x} = t,\quad t^2 - 28t + 27 = 0 \Rightarrow t = 1, 27$

If $t = 1 \Rightarrow \sin 2x + 2\cos^2 x = 0$

$$2\cos x(\sin x + \cos x) = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

If $t = 27$

$\Rightarrow \sin 2x + 2\cos^2 x = 3$ (Not possible)

$(\sin 2\alpha - \cos 2\alpha)^2 + 8\sin 4\alpha = 1 + 7\sin 4\alpha = 1$ (at $\alpha = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$)

18. $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$$= 5 + \operatorname{cosec}^2\theta + \sec^2\theta$$
$$= 7 + \tan^2\theta + \cot^2\theta$$
$$\geq 9$$

19. $\tan 20° + \tan 40° + \tan 80° - \tan 60°$

$$= \frac{\sin 20°\cos 80° + \sin 80°\cos 20°}{\cos 20°\cos 80°} + \frac{\sin 40°\cos 60° - \sin 60°\cos 40°}{\cos 40°\cos 60°}$$

$$= \frac{\sin 100°}{\cos 20° \cos 80°} - \frac{\sin 20°}{\cos 40° \cos 60°}$$

$$= \frac{\sin 80°}{\cos 20° \cos 80°} - \frac{2 \sin 20°}{\cos 40°}$$

$$= \frac{\sin 80° \cos 40° - \sin 40° \cos 80°}{\cos 20° \cos 40° \cos 80°} = \frac{\sin 40°}{\cos 20° \cos 40° \cos 80°}$$

$$= \frac{8 \sin 40° \sin 20°}{\sin(8 \times 20°)} = 8 \sin 40°$$

20. $1 + \cos 10x \cos 6x = 2\cos^2 8x + \sin^2 8x$

$2 + \cos 16x + \cos 4x = 2(1 + \cos 16x) + 1 - \cos 16x$

$\Rightarrow \cos 4x = 1$

$x = \frac{n\pi}{2} \quad (n = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots)$

If $360° < k < 540°$

$\Rightarrow k = 450° \quad (n = 5)$

21. $\cos 20° + 2\sin^2 55° = 1 + \sqrt{2} \sin k°$

$= \cos 20° + 1 - \cos 110°$

$= 1 + \cos 20° + \sin 20°$

$= 1 + \sqrt{2} \sin(45° + 20°)$

$\Rightarrow k = 65$

23. $\tan 19x = \frac{\cos 96° + \cos 6°}{\cos 96° - \cos 6°} = -\frac{2 \cos 51° \cos 45°}{2 \sin 51° \sin 45°} = -\cot 51° = \tan 141°$

$\Rightarrow 19x = 180° n + 141$

24. $\frac{2 \sin 40° + \sin 20°}{\cos 20° \cos 30°} = \frac{2 \sin (60° - 20°) + \sin 20°}{\cos 20° \cos 30°}$

25. $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} - 1 = \frac{\sin \frac{3\pi}{7}}{\sin \frac{\pi}{7}} \cdot \cos \frac{4\pi}{7} - 1 = \frac{-3}{2}$

26. $\frac{k}{2}(\cos 2A - \cos 3A) = \frac{11}{8}$

$\frac{k}{2}[2\cos^2 A - 1 - 4\cos^3 A + 3\cos A] = \frac{11}{8}$

$\Rightarrow k = 4$

27. $3\sin^2 x + 4\cos^2 x = 3 + \cos^2 x$

28. $\tan\alpha + \tan\beta = 12$

$\tan\alpha \cdot \tan\beta = -3$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=3$$

29. $\dfrac{\cos 24^\circ \cos 33^\circ}{2\sin 33^\circ \sin^2 57^\circ}+\left(\dfrac{\sin 18^\circ \cos 9^\circ}{\sin 9^\circ}-\cos 18^\circ\right)$

$$\frac{\cos 24^\circ \cos 33^\circ}{\sin 57^\circ \cos 24^\circ}+\frac{\sin 9^\circ}{\sin 9^\circ}=2$$

30. $\tan\theta\left(\dfrac{1+\cos 2\theta}{\cos 2\theta}\right)\left(\dfrac{1+\cos 4\theta}{\cos 4\theta}\right)\left(\dfrac{1+\cos 8\theta}{\cos 8\theta}\right)$

$$\frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)\left(\frac{2\cos^2 2\theta}{\cos 4\theta}\right)\left(\frac{2\cos^2 4\theta}{\cos 8\theta}\right)=\frac{8\sin\theta\cos\theta\cos 2\theta\cos 4\theta}{\cos 8\theta}=\frac{\sin 8\theta}{\cos 8\theta}=\tan 8\theta$$

31. $y=\sin^2 x+\cos^2 x+\tan^2 x+\cot^2 x+\operatorname{cosec}^2 x+\sec^2 x+6$

$$y=9+2(\tan^2 x+\cot^2 x)\geq 13$$

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Chapter 23 – Trigonometric Equations

23 TRIGONOMETRIC EQUATIONS

Exercise-1 : Single Choice Problems

1. $\tan^2 x - \sec^2 y = \frac{5a}{6} - 3 = -2 - a^2 \quad \Rightarrow 6a^2 + 5a - 6 = 0$

2. $[\tan(x+y) - \cot(x+y)]^2 + (x+1)^2 = 0$

$\Rightarrow \quad x = -1 \quad \text{and} \quad \tan^2(x+y) = 1$

$$x + y = n\pi \pm \frac{\pi}{4}$$

3.
$$\sin x + \cos x = 1$$
$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$$

4. $\sin^2(\sin x) - 3\sin(\sin x) + 2 = 0$

$\{\sin(\sin x) - 2\}\{\sin(\sin x) - 1\} = 0$

Equation has no solution.

5. $\tan 2x = \tan 6x \quad \Rightarrow \quad \sin 4x = 0$

$$4x = \pi, 2\pi, 3\pi, \ldots\ldots, 11\pi$$
$$x = \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \ldots\ldots, \frac{11\pi}{4}$$

But $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ are rejected. So number of solutions = 5.

6. $3\sin^2 x - 6\sin x - \sin x + 2 = 0$

$(3\sin x - 1)(\sin x - 2) = 0$

$\sin x \neq 2$, then

$$\sin x = \frac{1}{3}$$

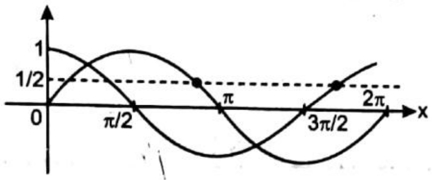
$\sin x = \frac{1}{3}$ has 6 solutions for $x \in [0, 5\pi]$

7. $$\cos\theta + \cos 2\theta = -1$$

$\Rightarrow$ $$2\cos^2\theta + \cos\theta = 0$$

$\Rightarrow$ $$\cos\theta = 0 \text{ or } \cos\theta = -\frac{1}{2}$$

$\Rightarrow$ $$\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$$

8. 

in $[0, 2\pi)$ max. $(\sin x, \cos x) = \frac{1}{2}$ has two solutions.

9. $(\cot^2 x + 2\sqrt{3}\cot x + 3) + (\cot^2 x + 1) + (4\operatorname{cosec} x + 4) = 0$

$(\cot x + \sqrt{3})^2 + (\operatorname{cosec} x + 2)^2 = 0$

$\Rightarrow$ $$\cot x = -\sqrt{3} \quad \text{and} \quad \operatorname{cosec} x = -2$$

10. $\sin^2 x = \sin^2 3x$

$\Rightarrow$ $3x = n\pi \pm x$

$$x = \frac{n\pi}{4},\ x = \frac{n\pi}{2}, \text{ hence general solution is } \frac{n\pi}{4}.$$

11. $\sin x > 0$

$\Rightarrow$ $$8\sin^2 x\cos^2 x = 1$$

$\Rightarrow$ $$2\sin^2 2x = 1$$

$\Rightarrow$ $$\cos 4x = 0$$

$$x = (2n+1)\frac{\pi}{8} \quad (n \in I)$$

12. $\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 5$

$\Rightarrow$ $\cos x = 1 \cap \cos 2x = 1 \cap \cos 3x = 1 \cap \cos 4x = 1 \cap \cos 5x = 1$

$$x = 2n\pi \cap x = n\pi \cap x = \frac{2n\pi}{3} \cap x = \frac{2n\pi}{4} \cap x = \frac{2n\pi}{5}$$

$\Rightarrow$ $x = 2n\pi$

13. $(2\sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$

$\Rightarrow$ $\sin^2 x = \frac{1}{2} \cap \tan^2 x = 1 \Rightarrow x = n\pi \pm \frac{\pi}{4}$

14. $$\cos^3 3x + \cos^3 5x = (2\cos 4x \cos x)^3$$
$$\cos^3 3x + \cos^3 5x = (\cos 5x + \cos 3x)^3$$
$$\Rightarrow \quad 3\cos 5x \cos 3x(\cos 5x + \cos 3x) = 0$$
$$\Rightarrow \quad \cos 5x \cos 3x \cdot \cos 4x \cos x = 0$$

15. $\sin^{100} x = 1 + \cos^{100} x \;\Rightarrow\; \sin^{100} x = 1$ and $\cos^{300} x = 0$

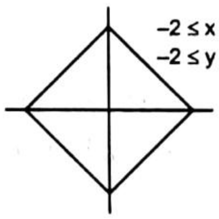
16. $\sin\theta \le 1$ and $\sec^2 4\theta \ge 1 \Rightarrow \sin\theta = \sec 4\theta = 1;\; \theta = \frac{\pi}{2}$

17. $(4\sin^2 x + \operatorname{cosec}^2 x) + (\tan^2 x + \cot^2 x) = 6$
$$(2\sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$$
$$\Rightarrow \quad 2\sin x = \operatorname{cosec} x \quad \text{and} \quad \tan x = \cot x$$

18. $\sin^4\theta - 2\sin^2\theta + 1 = 2$
$(\sin^2\theta - 1)^2 = 2 = \cos^4\theta$ (not possible)

19. $\cos(P\sin x) = \sin(P\cos x)$
$$\cos(P\sin x) = \cos\left(\frac{\pi}{2} - P\cos x\right)$$
$$P\sin x + P\cos x = 2n\pi + \frac{\pi}{2}$$
$$P\sin x - P\cos x = 2n\pi - \frac{\pi}{2}$$

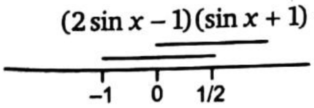
20. $|x| + |y| = 2$
$$\sin\left(\frac{\pi x^2}{3}\right) = 1$$
$$\frac{\pi x^2}{3} = \frac{\pi}{2}$$
$$x = \pm\sqrt{\frac{3}{2}}$$



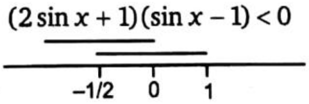
$-2 \le x \le 2$
$-2 \le y \le 2$

21. $x \in \left(-\frac{\pi}{2}, \pi\right)$
$$\cos 2x > |\sin x|$$

$\sin x \ge 0$
$$1 - 2\sin^2 x - \sin x > 0$$
$$2\sin^2 x + 2\sin x - \sin x - 1 < 0$$
$$(2\sin x - 1)(\sin x + 1) < 0$$



$\sin x < 0$
$$2\sin^2 x - 2\sin x + \sin x - 1 < 0$$
$$(2\sin x + 1)(\sin x - 1) < 0$$



$$-\frac{1}{2} < \sin x \le \frac{1}{2}$$

$$\left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$$

22. $\sin^4 x + \cos^4 x = \sin x \cos x$

$1 - 2\sin^2 x \cos^1 x = \sin x \cos x$

$2y^2 + y - 1 = 0$

$(2y-1)(y+1) = 0$

$y = \frac{1}{2}$ $\qquad$ $y = -1$

$2\sin x \cos x = 1$ $\qquad$ $\sin x \cos x \neq -1$

$\sin 2x = 1$

$2x = \frac{\pi}{2}, \frac{5\pi}{2}$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

23. $\sin\frac{5x}{2} = 1 \cap \sin\frac{x}{2} = -1$

24. $\cos 2\theta = \sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{3} \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$

25. $b\sin\theta = -c - a\cos\theta$

$b^2(1-\cos 2\theta) = c^2 + a^2\cos 2\theta - 2ac\cos\theta$

$\Rightarrow \quad (a^2+b^2)\cos 2\theta - 2ac\cos\theta + (c^2-b^2) = 0$

$$\cos\alpha \cdot \cos\beta = \frac{c^2-b^2}{a^2+b^2} \qquad \text{...(1)}$$

$a^2(1-\sin 2\theta) = c^2 + b^2\sin 2\theta - 2bc\sin\theta$

$(a^2+b^2)\sin 2\theta - 2bc\sin\theta + (c^2-a^2) = 0$

$$\sin\alpha \cdot \sin\beta = \frac{c^2-a^2}{a^2+b^2} \qquad \text{...(2)}$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{a^2-b^2}{a^2+b^2}$

Exercise-2 : One or More than One Answer is/are Correct

1. $2\cos^2\theta + 2\sqrt{2}\cos\theta - 3 = 0$

$$(\sqrt{2}\cos\theta + 1)^2 = 4 \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \text{ or } \frac{-3}{\sqrt{2}} \text{ (Not possible)}$$

3. $$4\sin 3x + 5 \geq 4\cos 2x + 5\sin x$$

$\Rightarrow \quad (\sin x - 1)(4\sin x + 1)^2 \leq 0 \; \forall \; x \in R$

4. $4\cos x(2 - 3\sin^2 x) + \cos 2x + 1 = 0$

$\cos x(3\cos x + 2)(2\cos x - 1) = 0$

Least difference $= \dfrac{\pi}{6}$

5. $\cos x \cos 6x = -1$

Case-1 : $\cos x = 1$ and $\cos 6x = -1$

Not possible

Case-2 : $\cos 6x = 1$ and $\cos x = -1$

$\Rightarrow \quad x = (2n-1)\pi, \; (n \in I)$

7. $$2k = \sin^2 2x - 2\sin 2x - 2$$

Let $\sin 2x = t \qquad t \in [-1, 1]$

$$2k = t^2 - 2t - 2 \quad \Rightarrow \quad k \in \left[-\frac{3}{2}, \frac{1}{2}\right]$$

8. $f(\theta) = \left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta - \cos\frac{5\pi}{8}\right)\left(\cos\theta - \cos\frac{7\pi}{8}\right) = \cos^4\theta - \cos^2\theta + \frac{1}{8}$

9. $\dfrac{4\sin^2 x\cos^2 x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4\cos^2 x - 4\sin^2 x\cos^2 x} = \tan^4 x = \dfrac{1}{9}$

$\Rightarrow \quad \tan x = \pm\dfrac{1}{\sqrt{3}}$

10. $\tan\theta(1 - \sin^2\theta) + \cot\theta(1 - \cos^2\theta) + 1 + \sin 2\theta = 0 \Rightarrow \sin 2\theta = -\dfrac{1}{2}$

11. $2\left(\dfrac{1}{2}\sin\theta + \dfrac{\sqrt{3}}{2}\cos\theta\right) = -(x-3)^2 - 2$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

1. $h(x) = f^2(x) + g^2(x) = 2 + 2\sin 4x$

$\Rightarrow \quad 8\cos 4x \geq 0$

$\Rightarrow \quad \cos 4x \geq 0$

Longest interval $= \dfrac{\pi}{4}$

2. $2 + 2\sin 4x = 4$

$\Rightarrow \quad \sin 4x = 1$

$\Rightarrow \quad x = (4n+1)\dfrac{\pi}{8}$

3. $\sin 3x + \cos x = \cos 3x + \sin x$

$\Rightarrow \quad \sin 3x - \sin x = \cos 3x - \cos x$

$\Rightarrow \quad 2\sin x \cos 2x = -2\sin 2x \sin x$

$\Rightarrow \quad \sin x = 0 \quad$ or $\quad \tan 2x = -1$

$\Rightarrow \quad x = 0, \pi \quad$ or $\quad x = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}$

Exercise-4 : Matching Type Problems

1. (A) $\cos^2 x = \left(\dfrac{1}{5} - \sin x\right)^2$

$\Rightarrow \quad (5\sin x - 4)(5\sin x + 3) = 0$

$\Rightarrow \quad \sin x = \dfrac{4}{5} \quad$ or $\quad -\dfrac{3}{5}$

(B) $\cot\dfrac{\theta}{2} = 1 + \cot\theta$

$\Rightarrow \quad 2\cos^2\dfrac{\theta}{2} = \cos\theta + \sin\theta$

$\Rightarrow \quad \sin\theta = 1 \;\Rightarrow\; \theta = -\dfrac{3\pi}{2}, \dfrac{\pi}{2}$

(C) $f(x) = -\sin^4 x + 8\sin^2 x + 2$

$\Rightarrow \quad f(x) \in [2, 9]$

(D) $\log_2 \dfrac{(2x^2 + 5x + 27)}{(2x-1)^2} \geq 0 \qquad \left(x > \dfrac{1}{2}\right)$

$\Rightarrow \quad 2x^2 - 9x - 26 \leq 0$

$\Rightarrow \quad -2 \le x \le \frac{13}{2}$

2. (A) $\sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$

(B) $f'(x) = \cos x + \sin x - K$

$\Rightarrow \quad k \ge \sqrt{2}$

(C) $|x^2 - 1| \le 1$ and $|2x^2 - 5| \le 1$

$\Rightarrow \quad x^2 = 2$

(D) $\sin x + \sin y = \sin(x + y)$

$$\Rightarrow \quad 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \quad \sin\left(\frac{x+y}{2}\right) = 0 \quad \text{or} \quad \cos\left(\frac{x-y}{2}\right) = \cos\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \quad x + y = 2n\pi \quad \text{or} \quad x = \frac{n\pi}{2},\ y = \frac{n\pi}{2}$$

if $x = 0,\ y = \pm 1$

if $x = \frac{1}{2},\ y = -\frac{1}{2}$

if $x = -\frac{1}{2},\ y = \frac{1}{2}$

if $y = 0,\ x = \pm 1$

Exercise-5 : Subjective Type Problems

1. Let $\sin x - 1 = a,\ \cos x - 1 = b,\ \sin x = c$

$\Rightarrow \quad a^3 + b^3 + c^3 = (a + b + c)^3$

$\Rightarrow \quad a + b = 0$ or $b + c = 0$ or $c + a = 0$

$\sin x + \cos x = 2$ or $\sin x + \cos x = 1$ or $\sin x = \frac{1}{2}$

$\Rightarrow \quad$ Total solution $= 5$

2. $\sin y - 2014 \cos y = 1$

$\Rightarrow \quad y = \frac{\pi}{2}$

3. $\frac{2\sin 6x}{\sin x - 1} < 0$

$\Rightarrow \quad \sin 6x > 0$

$\Rightarrow \quad x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$$1+\tan^2 x - 2\sqrt{2}\tan x \le 0$$

$$\Rightarrow \quad x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \Rightarrow x \in \left[\frac{\pi}{8}, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{8}, \frac{3\pi}{8}\right]$$

4. $\sin^4 x - 4\sin^2 x + (2+k) = 0$

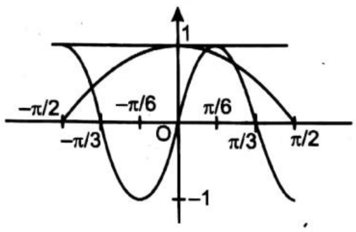
Let $\sin^2 x = t \qquad t \in [0, 1]$

$$t^2 - 4t + (2+k) = 0$$

$$f(0)\,f(1) \le 0$$

$$(k+2)(k-1) \le 0 \Rightarrow -2 \le k \le 1$$

5.



6. $2\sin^2 x + \sin^2 2x = 2$

$$2\sin^4 x - 3\sin^2 x + 1 = 0 \Rightarrow (2\sin^2 x - 1)(\sin^2 x - 1) = 0$$

$$\sin 2x + \cos 2x = \tan x$$

$$2\tan x + 1 - \tan^2 x = \tan x(1+\tan^2 x)$$

$$\Rightarrow \quad \tan^3 x + \tan^2 x - \tan x - 1 = 0 \Rightarrow (1+\tan x)(\tan^2 x - 1) = 0$$

$$2\cos^2 x + \sin x \le 2$$

$$2\sin^2 x - \sin x \ge 0$$

$$\sin x(2\sin x - 1) \ge 0$$

7. $(3\cot\theta + 1)(\cot\theta + 3) = 0$

$\cot\theta = -\frac{1}{3}$ and $\cot\theta = -3$

$\theta = \alpha, \pi + \alpha \qquad \theta = \frac{\pi}{2} - \alpha, \pi + \frac{\pi}{2} - \alpha$

8. $(8\cos 4\theta - 3)(\cot\theta - \tan\theta)^2 = 12$

$$8(2\cos^2 2\theta - 1) - 3\left(\frac{4\cos^2 2\theta}{\sin^2 2\theta}\right) = 12$$

$$16\cos^4 2\theta - 8\cos^2 2\theta - 3 = 0$$

$$\Rightarrow \quad (4\cos^2 2\theta - 3)(4\cos^2 2\theta + 1) = 0$$

$$\Rightarrow \quad \cos 2\theta = \pm\frac{\sqrt{3}}{2}$$

9. $2\sin^2 x + 4\sin^2 x\cos^2 x = 2$

$2\sin^4 x - 3\sin^2 x + 1 = 0 \quad \Rightarrow (\sin^2 x - 1)(2\sin^2 x - 1) = 0$

$$\sin x = \pm\frac{1}{\sqrt{2}}, \pm 1$$

$$\sin 2x + \cos 2x = \tan x$$

$$\frac{2\tan x}{1+\tan^2 x} + \frac{1-\tan^2 x}{1+\tan^2 x} = \tan x$$

$$\Rightarrow \quad \tan^3 x + \tan^2 x - \tan x - 1 = 0$$

$$(\tan x + 1)^2(\tan x - 1) = 0$$

$$2\cos^2 x + \sin x \le 2$$

$$2\sin^2 x - \sin x \ge 0$$

$$\sin x(2\sin x - 1) \ge 0$$

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Chapter 24 – Solution of Triangles

24 SOLUTION OF TRIANGLES

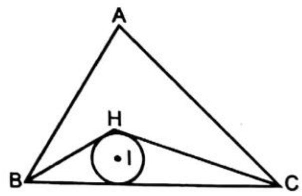
Exercise-1 : Single Choice Problems

1. $\dfrac{\cot A+\cot B}{\cot C}=\dfrac{\cos A\sin B+\cos B\sin A}{(\sin A\sin B)\dfrac{\cos C}{\sin C}}=\dfrac{\sin^2 C}{(\sin A\sin B)\cos C}=\dfrac{c^2}{ab\cdot\dfrac{a^2+b^2-c^2}{2ab}}$

$$=\frac{2c^2}{\left(\frac{17}{9}-1\right)c^2}=\frac{18}{8}$$

2. $\angle BIC=\dfrac{\pi}{2}+\left(\dfrac{\pi-A}{2}\right)$

$=\dfrac{\pi}{2}+\left(\dfrac{B+C}{2}\right)$



3. $\dfrac{1}{64}[(2R\cos A)^2+a^2][(2R\cos B)^2+b^2][(2R\cos C)^2+c^2]$

$\dfrac{1}{64}[(2R\cos A)^2+(2R\sin A)^2][(2R\cos B)^2+(2R\sin B)^2][(2R\cos C)^2+(2R\sin C)^2]$

$=R^6$

4. $B=60°$

$2\sin^2 B=3\sin^2 C\Rightarrow \sin C=\dfrac{1}{\sqrt{2}}\Rightarrow C=45°$

5. $\tan\dfrac{A}{2}\tan\dfrac{C}{2}=\dfrac{1}{3}$

$\dfrac{s-b}{s}=\dfrac{1}{3}\Rightarrow b=\dfrac{2}{3}s\Rightarrow \dfrac{a+c}{2}=b\Rightarrow b\geq 2$ (A.M. ≥ G.M.)

6. $\cos A\cos B\cos C\sum\dfrac{a}{\cos A}=2R\cos A\cos B\cos C\sum\tan A=2R\cos A\cdot\cos B\cdot\cos C\cdot\prod(\tan A)$

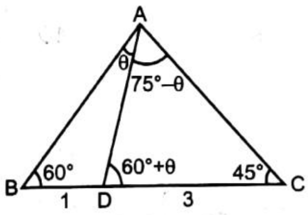
$$= 2R\cos A\cdot\cos B\cdot\cos C\cdot\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C} = 2R\sin A\sin B\sin C$$

8. In ΔBAD, $\dfrac{BD}{\sin\theta} = \dfrac{AD}{\sin 60°}$

In ΔCAD, $\dfrac{CD}{\sin(75°-\theta)} = \dfrac{AD}{\sin 45°}$

$\Rightarrow \quad \dfrac{BD}{\sin\theta}\sin 60° = \dfrac{CD\sin 45°}{\sin(75°-\theta)}$

$\Rightarrow \quad \dfrac{\sin\theta}{\sin(75°-\theta)} = \dfrac{BD}{CD}\dfrac{\sin 60°}{\sin 45°} = \dfrac{1}{\sqrt{6}}$



9. Length of angle bisector $AD = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$

Length of angle bisector $BE = \dfrac{2ac}{a+c}\cos\dfrac{B}{2}$

Length of angle bisector $CF = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$

$$\text{H.M.} = \frac{3}{\dfrac{b+c}{2bc}+\dfrac{a+c}{2ac}+\dfrac{a+b}{2ab}} = \frac{3}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}$$

10. $2b = a + c$

$2\sin B = \sin A + \sin C$

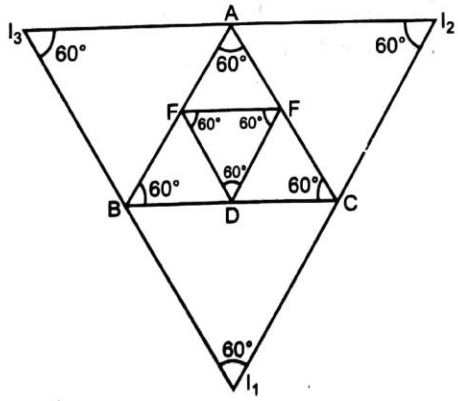
$$2\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right) = 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right) \Rightarrow \sin\frac{B}{2} = \frac{1}{2\sqrt{2}}$$

11. $2\cos\left(\dfrac{B-C}{2}\right) = \dfrac{b+c}{a} = \dfrac{\sin B+\sin C}{\sin A}$

$\Rightarrow \quad \sin\dfrac{A}{2} = \dfrac{1}{2} \quad \Rightarrow \quad \angle A = 60°$

12. $\cos A = \dfrac{4+c^2-1}{4c} = \dfrac{1}{4}\left(c+\dfrac{3}{c}\right) \geq \dfrac{\sqrt{3}}{2}$

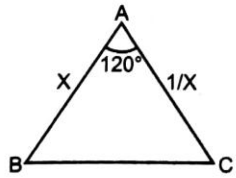
13.



If ΔABC is an equilateral triangle then ΔDEF and $\Delta I_1I_2I_3$ are also equilateral triangle

Side of $\Delta DEF = 1$ unit $\Rightarrow Ar(\Delta DEF) = \frac{\sqrt{3}}{4}$

14. $$AD = \frac{2x \cdot \frac{1}{x}}{x + \frac{1}{x}} \cos\frac{\pi}{3} = \frac{1}{x + \frac{1}{x}}$$



$$AD_{\max} = \frac{1}{2}$$

15. $r = \frac{\sqrt{3}a}{6}, R = \frac{\sqrt{3}a}{3}, r_1 = \frac{\sqrt{3}a}{2} \Rightarrow r, R, r_1$ are in A.P.

16. $\sin(B+C)\sin(B-C) = \sin(A+B)\sin(A-B)$

$$\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$

$\Rightarrow$ $$2\sin^2 B = \sin^2 A + \sin^2 C$$

$\Rightarrow$ $$2b^2 = a^2 + c^2$$ (Using sine rule)

17. $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(\pi - C)$$

$\Rightarrow$ $$\tan C = \frac{7}{4} \Rightarrow \sin C = \frac{7}{\sqrt{65}}$$

Using sine rule

$$R = \frac{c}{2\sin C} = \frac{65}{14}$$

18. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

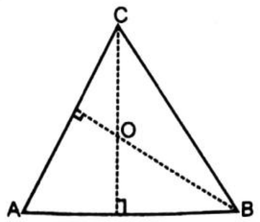
$$\frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \frac{(a+b+c)^2 - 2(ab+bc+ac)}{2abc}$$

19. $\frac{a+c}{b} + \frac{b+c}{a} = \frac{a^2+b^2+ac+bc}{ab} = \frac{c^2(a+b+c)}{abc} = \frac{c^2(2s)}{4R\Delta} = \frac{2R}{r} = \frac{c}{r}$

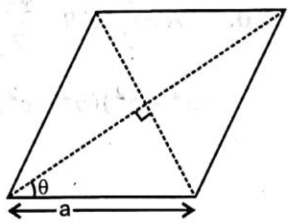
20. $a^2(\sin B - 1) = b^2 + c^2 - a^2 = 2bc\cos A \Rightarrow \cos A < 0$

21. $2R' = \frac{a}{\sin(\pi - A)} = \frac{a}{\sin A} = 2R$

$\Rightarrow$ $$R' = R$$



22. $a=\sqrt{d_1 d_2}$

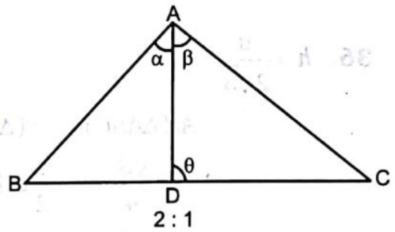


$\frac{d_1}{2}=a\cos\theta,\ \frac{d_2}{2}=a\sin\theta$

$1=4\sin\theta\cos\theta$

$\Rightarrow \sin 2\theta=\frac{1}{2}\Rightarrow 2\theta=30^\circ$

23. $\because \quad (m+n)\cot\theta=m\cot\alpha-n\cot\beta$



$\therefore \quad (2+1)\cot\theta=2\cot\alpha-\cot\beta$

Put $\cot\theta=\frac{1}{3},\cot\beta=\cot\left(\frac{\pi}{2}-\alpha\right)=\tan\alpha$

We have $1=\frac{2}{\tan\alpha}-\tan\alpha \Rightarrow \tan^2\alpha+\tan\alpha-2=0$

$\therefore \quad \tan\alpha=1 \qquad \therefore \quad \alpha=45^\circ$

24. Circumradius of equilateral $\Delta, R=\frac{l}{2\sin 60^\circ}=\frac{l}{\sqrt{3}}$

Diagonal of square $=2R\Rightarrow a\sqrt{2}=2R \quad \therefore a=R\sqrt{2}=\frac{l\sqrt{2}}{\sqrt{3}} \quad \therefore$ Area of square $=\frac{2l^2}{3}$

25. $\cos\theta=\frac{2^2+(\sqrt{6})^2-(\sqrt{3}+1)^2}{4\sqrt{6}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

26. If a,b,c are in A.P.

$\Rightarrow \quad 2\sin B=\sin A+\sin C\Rightarrow \sin\frac{B}{2}=\frac{1}{4}$

$$\frac{s}{r}=\frac{6\cos\frac{B}{2}}{1-2\sin\frac{B}{2}}=3\sqrt{15}$$

27. $\cos(A-B)=\frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}=\frac{31}{32}\Rightarrow \tan\left(\frac{A-B}{2}\right)=\frac{1}{3\sqrt{7}}$

$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2} \quad \Rightarrow \cos C=\frac{1}{8}$

$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{1}{8} \quad \Rightarrow c=6$

28. $(b+c)\cos(B+C)+(c+a)\cos(C+A)+(a+b)\cos(A+B)$

$=-[(b\cos A+a\cos B)+(c\cos A+a\cos C)+(c\cos B+b\cos C)]=-[a+b+c]=-30$

30. $\angle A = \frac{\pi}{7}, \angle B = \frac{2\pi}{7}, \angle C = \frac{4\pi}{7}$

$$(a^2 - b^2)(b^2 - c^2)(c^2 - a^2) = a^2b^2c^2\left(1 - \frac{b^2}{a^2}\right)\left(1 - \frac{c^2}{b^2}\right)\left(1 - \frac{a^2}{c^2}\right)$$

$$= a^2b^2c^2\left(1 - \frac{\sin^2\frac{2\pi}{7}}{\sin^2\frac{\pi}{7}}\right)\left(1 - \frac{\sin^2\frac{4\pi}{7}}{\sin^2\frac{2\pi}{7}}\right)\left(1 - \frac{\sin^2\frac{\pi}{7}}{\sin^2\frac{4\pi}{7}}\right) = a^2b^2c^2$$

36. $h = \frac{a}{2\sqrt{3}}$

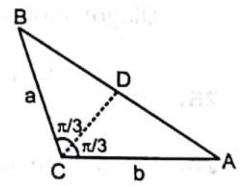
$$Ar(\Delta ABC) = Ar(\Delta APB) + Ar(\Delta BPC) + Ar(\Delta APC)$$

$$\frac{\sqrt{3}}{4}a^2 = \frac{1}{2}a(h + h_1 + h_2) \quad \Rightarrow \quad h_1 + h_2 = \frac{a}{\sqrt{3}}$$

37. $\cos 60° = \frac{6^2 + 7^2 - x^2}{2 \times 6 \times 7} \Rightarrow x = \sqrt{43}$

39. $CD = \frac{2ab}{a+b}\cos\frac{\pi}{3}$

$= \frac{ab}{a+b}$



42. $a + b + c = 48$

$a = 20$

$b + c = 28$

$\Rightarrow$ $a + b > c, a + c > b$

$\Rightarrow$ $20 + b > 28 - b$, $20 + c > 28 - c$

$\Rightarrow$ $b > 4, c > 4$

43. In an equilateral triangle

$a = b = c$

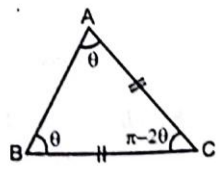
44. $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$

$$= \frac{2s(2s-2a)(2s-2b)(2s-2c)}{4b^2c^2} = 4\left(\frac{2(s-a)}{bc}\right)\left(\frac{(s-b)(s-c)}{bc}\right) = 4\sin^2\frac{A}{2}\cos^2\frac{A}{2} = \sin^2 A$$

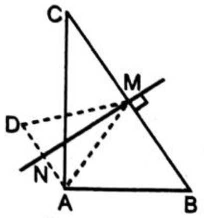
45. $R = 4r$

$R = 4\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$1 = 16\sin^2\frac{\theta}{2}\cdot\cos\theta = 8(1 - \cos\theta)\cos\theta$



46. $\Delta DMN \cong \Delta AMN \Rightarrow DM = AM$



47. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^2 + c^3 - 3abc)$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

48. $A = \frac{\pi}{7}, B = \frac{2\pi}{7}, C = \frac{4\pi}{7}$

$$\left(1-\frac{b^2}{a^2}\right)\left(1-\frac{c^2}{b^2}\right)\left(1-\frac{a^2}{c^2}\right) = \lambda$$

$$\left(1-\frac{\sin^2\frac{2\pi}{7}}{\sin^2\frac{\pi}{7}}\right)\left(1-\frac{\sin^2\frac{4\pi}{7}}{\sin^2\frac{2\pi}{7}}\right)\left(1-\frac{\sin^2\frac{\pi}{7}}{\sin^2\frac{4\pi}{7}}\right) = \lambda$$

$$\left(\frac{\sin^2\frac{\pi}{7}-\sin^2\frac{2\pi}{7}}{\sin^2\frac{\pi}{7}}\right)\left(\frac{\sin^2\frac{2\pi}{7}-\sin^2\frac{4\pi}{7}}{\sin^2\frac{2\pi}{7}}\right)\left(\frac{\sin^2\frac{4\pi}{7}-\sin^2\frac{\pi}{7}}{\sin^2\frac{4\pi}{7}}\right) = \lambda$$

$\Rightarrow \quad \lambda = 1(\sin^2 A - \sin^2 B = \sin(A-B)\cdot\sin(A+B))$

49. $r_1 = \frac{\Delta}{s-a}, \quad r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$

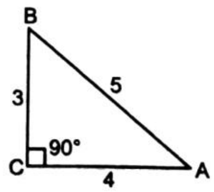
$$\frac{r_1 r_2 r_3}{r^3} = \frac{s^3}{(s-a)(s-b)(s-c)}$$

$$\frac{\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}}{3} \geq \left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{s}\right)$$

50. $\sin A = \frac{3}{5}$

$\sin B = \frac{4}{5}$

$\sin C = 1$



51. $\dfrac{r_1 + r_2}{1 + \cos C} = \dfrac{4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}{2\cos^2\frac{C}{2}}$

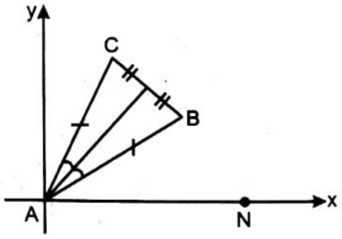
$$= \frac{2R\left(\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right)}{\cos\frac{C}{2}} = 2R$$

53. $\cos\theta = \dfrac{\sin^2\alpha + \cos^2\alpha - (1 + \sin\alpha\cos\alpha)}{2\sin\alpha\cos\alpha} = \dfrac{1}{2}$

55. Since we need to compute the radius of an escribed circle, we would be needing the length of all the sides of the given triangle ABC.

From the question, we already know $AB = AC = 5$.

For finding the length of side BC, let us draw a line AD which is the bisector of angle BAC, as shown in the figure below.



$$\angle BAD = \angle DAC = 15°$$

Therefore, $\sin 15° = \dfrac{BD}{AB} = \dfrac{BD}{5}$ and $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

Therefore, $BD = 5\sin 15° = \dfrac{5(\sqrt{3}-1)}{2\sqrt{2}}$

We also know that $BC = 2BD$

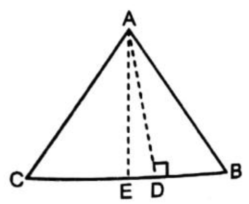
Therefore, $BC = \dfrac{5(\sqrt{3}-1)}{\sqrt{2}}$

Now, we know that the required radius

$$r_1 = s\tan\left(\frac{A}{2}\right) = \left(\frac{AB + BC + CA}{2}\right)\tan\left(\frac{A}{2}\right)$$

$$= \left(\frac{5 + \frac{5(\sqrt{3}-1)}{\sqrt{2}} + 5}{2}\right)(\tan 15°) = \left(\frac{10\sqrt{2} + 5\sqrt{3} - 5}{2\sqrt{2}}\right)(2 - \sqrt{3})$$

56. $ED = BE - BD = \dfrac{a}{2} - C\cos B$



$$= \frac{a}{2} - C\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

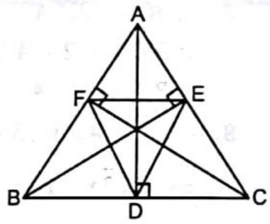
$$= \frac{b^2 - c^2}{2a}$$

57. $2R(\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C)$

$$=2R(\sin(A+B)\cos C+\cos A\cos B\sin C)$$

$$=R(2\sin A\sin B\sin C)=\frac{abc}{4R^2}=\frac{\Delta}{R}=\frac{rs}{R}$$

58. In ΔAFE, $\frac{b\cos A}{\sin B}=2R_1$



$\Rightarrow$ $R_1=R\cos A$

Similarly, $R_2=R\cos B$

and $R_3=R\cos C$

$R_1+R_2+R_3=R(\cos A+\cos B+\cos C)\le\frac{3}{2}R$

59. $\text{Ar}(\Delta ABC)=\text{Ar}(\Delta OAB)+\text{Ar}(\Delta OBC)+\text{Ar}(\Delta OAC)$

$$8=\frac{1}{2}R^2(\sin\alpha+\sin\beta+\sin\gamma)$$

$\Rightarrow$ $\sin\alpha+\sin\beta+\sin\gamma=\frac{4\pi}{5}$ $\left(\because R^2=\frac{20}{\pi}\right)$

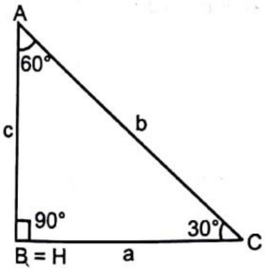
Exercise-2 : One or More than One Answer is/are Correct

1. $x^2-r(r_1r_2+r_2r_3+r_1r_3)x+(r_1r_2r_3-1)=0$

$$x^2-(r_1r_2r_3)x+(r_1r_2r_3-1)=0$$

$\Rightarrow$ Roots are 1 and $r_1r_2r_3-1$

2.



3. $R=2r, r=(s-a)\tan 30°=\frac{s}{3}\tan 30°\Rightarrow s$ is irrational $\Rightarrow\Delta$ is irrational

$$r_1=s\tan 30°=3r \text{ (rational)}$$

4. $D+E+F=\frac{\pi}{2}$

5. $a=4, b=8, \angle C=60°$

$$\cos C=\frac{1}{2}=\frac{a^2+b^2-c^2}{2ab}\Rightarrow c=4\sqrt{3}$$

6. If $\frac{r}{r_1} = \frac{r_2}{r_3} \Rightarrow \frac{s-a}{s} = \frac{s-c}{s-b}$

$\Rightarrow \quad a^2 + b^2 = c^2$

$\Rightarrow \quad \angle C = 90°$

7. $\angle BOC = 2\angle A$

$\angle BIC = \pi/2 + A/2$

$\angle BHC = \pi - A$

8. $\sqrt{3}x^2 - 4x + \sqrt{3} < 0$

$\Rightarrow \quad (\sqrt{3}x - 1)(x - \sqrt{3}) < 0 \qquad \Rightarrow \frac{1}{\sqrt{3}} < x < \sqrt{3}$

$30° < A, B < 60° \qquad \Rightarrow 60° < C < 120°$

9. $\cos 2\theta = 2\cos^2\theta - 1$

$$\frac{1}{\sqrt{2}} = 2\cos^2\frac{\pi}{8} - 1$$

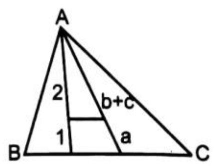
$$2\cos^2\frac{\pi}{8} = 1 + \frac{1}{\sqrt{2}}$$

$$\cos^2\frac{\pi}{8} = \frac{\sqrt{2}+1}{2\sqrt{2}}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \text{ then solve it}$$

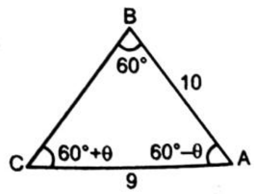
11. $(3\sin A + 4\cos B)^2 + (4\sin B + 3\cos A)^2 = 37; \quad 9 + 16 + 24\sin(A+B) = 37$

12. $\frac{b+c}{a} = \frac{2}{1}$



13. $\angle A, \angle B, \angle C$ A.P. $\Rightarrow \angle B = 60°$

$\cos 60° = \frac{a^2 + 10^2 - 9^2}{20a}$



14. $\Delta = \frac{1}{2}ab\sin C$

$\frac{a+b}{2} \geq \sqrt{ab} \quad \Rightarrow \quad \frac{\sin A + \sin B}{2} \geq \sqrt{\sin A \times \sin B}$

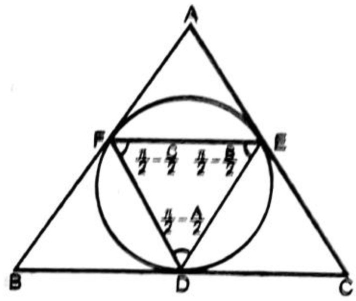
15. $3\cos A = \cos(B - C) - \cos(B + C) \Rightarrow 2\cos A = \cos(B - C) = -\cos(A + 2C)$

$2 = (\tan A \sin 2C - \cos 2C)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 2

2. $r' = 4r\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\sin\left(\frac{\pi}{4} - \frac{B}{4}\right)\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)$



$\frac{r'}{r} = 4\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\sin\left(\frac{\pi}{4} - \frac{B}{4}\right)\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)$

$= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1$

$r_1' = 4r\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\cos\left(\frac{\pi}{4} - \frac{B}{4}\right)\cos\left(\frac{\pi}{4} - \frac{C}{4}\right)$

$\frac{r_1'}{r} = 4\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\cos\left(\frac{\pi}{4} - \frac{B}{4}\right)\cos\left(\frac{\pi}{4} - \frac{C}{4}\right)$

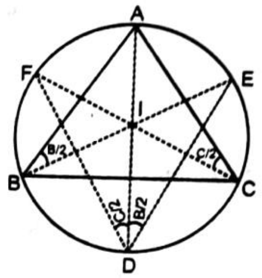
$= 1 - \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

Paragraph for Question Nos. 3 to 4

3. $Ar(\Delta DEF)$

$= 2R^2 \sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\sin\left(\frac{\pi}{2} - \frac{B}{2}\right)\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$= 2R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$



4. $\frac{Ar(\Delta ABC)}{Ar(\Delta DEF)} = \frac{2R^2 \sin A \sin B \sin C}{2R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le 1$

Paragraph for Question Nos. 5 to 6

Sol. $c/2 = R \Rightarrow c = 82$

$2r = a + b - c$

$\Rightarrow \quad a + b = 98 \quad \ldots(1)$

$a^2 + b^2 = c^2 = (82)^2 \quad \ldots(2)$

$\Rightarrow \quad a = 18, b = 80$

Paragraph for Question Nos. 7 to 8

Sol. $\angle A_1 = \frac{\pi}{2} - \frac{A}{2}$

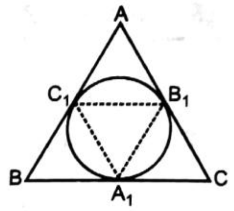
$$\angle A_2 = \frac{\pi}{2} - \frac{1}{2}(\angle A_1) = \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \frac{A}{2}\right)$$

$$= \frac{\pi}{4} + \frac{A}{4}$$

$$\angle A_3 = \frac{\pi}{2} - \frac{1}{2}(\angle A_2)$$

$$= \frac{3\pi}{8} - \frac{A}{8}$$

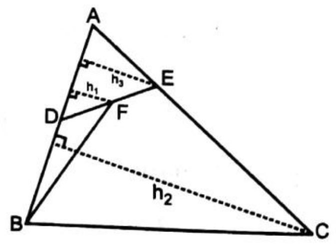
$$\angle A_n = \frac{\pi}{2}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\ldots\right) + \frac{(-1)^n A}{2^n}$$



Paragraph for Question Nos. 9 to 10

Sol. $\frac{\Delta_1}{\Delta} = \frac{\frac{1}{2} \times BD \times h_1}{\frac{1}{2} \times AB \times h_2} = (1-x)\frac{h_1}{h_2} \times \frac{h_3}{h_3} = (1-x)\,yz$

$$\frac{\Delta_2}{\Delta} = \frac{\frac{1}{2} \times EC \times h_4}{\frac{1}{2} \times AC \times h_5} = x(1-y)(1-z)$$



Paragraph for Question Nos. 11 to 13

Sol. $\log\left(1 + \frac{c}{a}\right) + \log a - \log b = \log 2$

$\Rightarrow \quad a + c = 2b$

$(c-a)x^2 + 2bx + (a+c) = 0$ has equal roots, then

$$a^2 + b^2 = c^2$$

Paragraph for Question Nos. 14 to 16

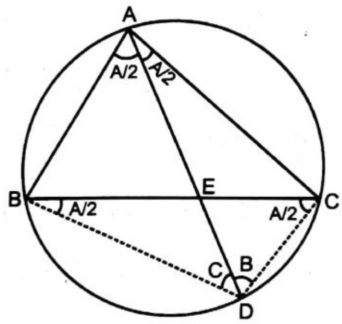
Sol. $\frac{BE}{\sin C} = \frac{ED}{\sin\frac{A}{2}}, \frac{EC}{\sin B} = \frac{ED}{\sin\frac{A}{2}}$

$\Rightarrow \quad BE + EC = a = \dfrac{ED}{\sin\dfrac{A}{2}}(\sin B + \sin C)$

$\Rightarrow \quad ED = \dfrac{a\sin\dfrac{A}{2}\times 2R}{b+c}$

$$\dfrac{l_a = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}}{\left(\dfrac{2bc}{b+c}\cos\dfrac{A}{2} + \dfrac{a\sin\dfrac{A}{2}\times 2R}{b+c}\right)}$$

$\Rightarrow \quad l_a = \dfrac{2\sin B\sin C}{2\sin B\sin C + 2\sin^2\dfrac{A}{2}} = \dfrac{\sin B\sin C}{\sin^2\left(B+\dfrac{A}{2}\right)}$



Exercise-4 : Matching Type Problems

2. (A) $3^0\{2^0 + 2^{-1} + 2^{-2} \ldots\ldots\ldots\infty\} = 1\{2\}$

$3^{-1}\{2^0 + 2^{-1} + 2^{-2} \ldots\ldots\ldots\infty\} = \dfrac{1}{3}\{2\}$

$3^{-2}\{2^0 + 2^{-1} + 2^{-2} \ldots\ldots\ldots\infty\} = \dfrac{1}{3}\{2\}$

$\vdots \qquad\qquad \vdots$

∞

Hence, $\dfrac{2\times 1}{1-\dfrac{1}{3}} = 3$

(B) $b^2 + c^2 - a^2 = 2bc\cos A = 54$

$bc\cos A = 27 = a^3 \Rightarrow a = 3$

$\dfrac{b^2+c^2}{9} = \dfrac{63}{9} = 7$

(C) Circumcentre of ΔABC is $(-1, 0)$.

Point A lie on the circle $(x+1)^2 + y^2 = 4 \Rightarrow x^2 + y^2 + 2x - 3 = 0$

(D) $(\cos\theta\sin\theta + 6) = 6(\sin\theta - \cos\theta) \Rightarrow 36 + \sin^2\theta\cos^2\theta + 12\sin\theta\cos\theta = 36(1 - 2\sin\theta\cos\theta)$

Let $\sin\theta\cos\theta = t$

$$t^2 + 84t = 0 \Rightarrow t = 0$$

If $\sin\theta = 0 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

If $\cos\theta = 0 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

3. $r_1r_2 + r_3r_2 + r_1r_3 = S^2 \Rightarrow S = 42$

$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r} \Rightarrow r = 8$

$$r = \frac{\Delta}{S} \Rightarrow \Delta = 336$$

4. Use $r = \frac{\Delta}{s}$, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-a}$

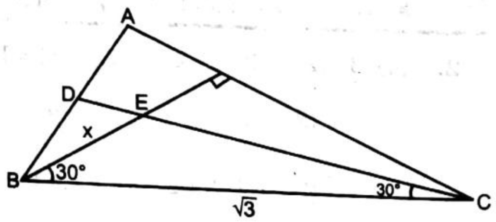
(C) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

and similarly r_1, r_2, r_3

Exercise-5 : Subjective Type Problems

2. $\angle O_1EO_2 = 90°$, E is the orthocentre of ΔO_1EO_2

$\frac{x}{\sin 30°} = \frac{\sqrt{3}}{\sin 120°}$; $x = 1$

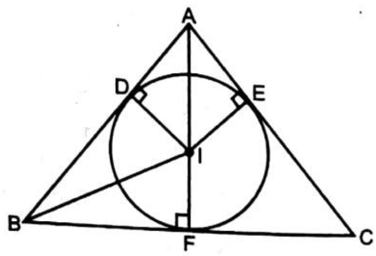


3. $\frac{1}{2}r(AD + AE) = 5$

$\frac{1}{2}r(BF + BD) = 10$

$$\Rightarrow \frac{BF + BD}{AD + AE} = 2 \Rightarrow \frac{r\cot\frac{B}{2} + r\cot\frac{B}{2}}{r\cot\frac{A}{2} + r\cot\frac{A}{2}} = 2$$

Applying C and D, $\frac{\cos\frac{C}{2}}{\sin\frac{A-B}{2}} = 3$



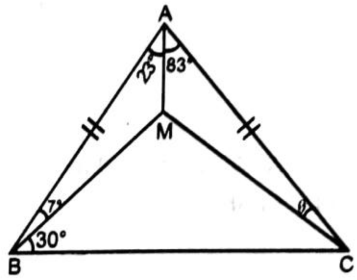
4. $\frac{\Delta_1\Delta_2\Delta_3}{\Delta^3} = \frac{(r_1r_2r_3)^2}{r^6} = \left(\frac{s}{s-a} \times \frac{s}{s-b} \times \frac{s}{s-c}\right)^2$

$\frac{(s-a) + (s-b) + (s-c)}{3} \geq [(s-a)(s-b)(s-c)]^{1/3}$

$\Rightarrow \quad \dfrac{s^3}{(s-a)(s-b)(s-c)} \geq 27$

Minimum value = 1

5. In ΔABM, $\quad \dfrac{AB}{\sin 150°} = \dfrac{AM}{\sin 7°}$



In ΔACM, $\dfrac{AC}{\sin(97°-\theta)} = \dfrac{AM}{\sin\theta}$

$\Rightarrow \quad \sin\theta = 2\sin 7° \sin(97°-\theta)$

$\Rightarrow \quad \sin\theta = \sin\theta - \cos(104°-\theta)$

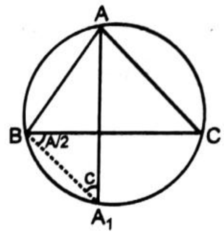
$\Rightarrow \quad \cos(104°-\theta) = 0$

$\Rightarrow \quad \theta = 14°$

6. $\dfrac{AH}{AD} + \dfrac{BH}{BE} + \dfrac{CH}{CF} = \dfrac{R}{\Delta}(a\cos A + b\cos B + c\cos C) = \dfrac{R^2}{\Delta}(\sin 2A + \sin 2B + \sin 2C)$

$= \dfrac{4R^2}{\Delta}\sin A \sin B \sin C = \dfrac{bc\sin A}{\Delta} = 2$

7. $\dfrac{c}{\sin C} = \dfrac{AA_1}{\sin\left(B + \dfrac{A}{2}\right)}$



$AA_1 \cos\dfrac{A}{2} = \sin B + \sin C \qquad (\because R = 1)$

$\Rightarrow \quad \dfrac{AA_1\cos\dfrac{A}{2} + BB_1\cos\dfrac{B}{2} + CC_1\cos\dfrac{C}{2}}{\sin A + \sin B + \sin C} = 2$

8. $ax^2 + bx + c = 0$ has equal roots, then

$$b^2 = 4ac \qquad \ldots(1)$$

$$\frac{\sin A}{\sin C} + \frac{\sin C}{\sin A} = \frac{a}{c} + \frac{c}{a} = \frac{a^2+c^2}{ac} = \frac{b^2 + 2ac\cos B}{ac}$$

$$= 4 + 2\cos B$$

9. $\cot\dfrac{A}{2}, \cot\dfrac{B}{2}, \cot\dfrac{C}{2}$ is AP

In ΔABC,

$\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cdot\cot\dfrac{B}{2}\cdot\cot\dfrac{C}{2}$

$\Rightarrow \quad \cot\dfrac{A}{2}\cdot\cot\dfrac{C}{2} = 3$

AM ≥ GM

$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2} \geq \sqrt{\cot\frac{A}{2}\cdot\cot\frac{C}{2}} \quad \Rightarrow \cot\frac{B}{2} \geq \sqrt{3}$$

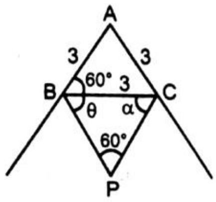
10. $(R^2 - 4Rr + 4r^2) + (4r^2 - 12r + 9) = 0$

$$(R - 2r)^2 + (2r - 3)^2 = 0$$

$\Rightarrow$ $r = \frac{3}{2}$; $R = 2r$

ΔABC is an equilateral triangle.

11. In ΔBCP,



$$\frac{3}{\sin 60°} = \frac{PC}{\sin\theta}$$

$$PC = 2\sqrt{3}\sin\theta$$

12. $b + c = \frac{2ab\cos C + 2\sqrt{3}ab\sin C}{2b} = \frac{(a^2 + b^2 - c^2) + 12}{2b}$

13. $R = 3, \Delta = 6$

$P_{\Delta DEF} = DE + EF + DF = R(\sin 2A + \sin 2B + \sin 2C)$

$= 4R\sin A\sin B\sin C$

$= 4R\left(\frac{b}{2R}\frac{c}{2R}\sin A\right) = \frac{1}{R}(2\Delta) = 4$

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Chapter 25 – Inverse Trigonometric Functions

25 Inverse Trigonometric Functions

Exercise-1 : Single Choice Problems

2. $(\cot^{-1} x)\left(\frac{\pi}{2} - \cot^{-1} x\right) + 2\cot^{-1} x - \frac{\pi}{2}\cot^{-1} x + 3\left(\frac{\pi}{2} - \tan^{-1} x\right) - 6 > 0$

$$-(\cot^{-1} x)^2 + 5\cot^{-1} x - 6 > 0$$

$$(\cot^{-1} x)^2 - 5(\cot^{-1} x) + 6 < 0$$

$$(\cot^{-1} x - 3)(\cot^{-1} x - 2) < 0$$

$$2 < \cot^{-1} x < 3$$

$$\cot 3 < x < \cot 2 \qquad (\because \cot^{-1} x \text{ is decreasing})$$

3. $1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3) = 1 + 2^2 + 1 + 3^2 = 15$

4. $\sum_{n=1}^{\alpha} \tan^{-1}\left(\frac{(n+1)^2 + (n+1) - ((n+1)^2 - (n+1))}{1 + (n+1)^4 - (n+1)^2}\right)$

5. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$$\frac{\pi}{2} - 2\tan^{-1}\sqrt{\cos\alpha} = x$$

$$\frac{\pi}{4} - \frac{x}{2} = \tan^{-1}\sqrt{\cos\alpha}$$

$$\sqrt{\cos\alpha} = \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

$\Rightarrow$
$$\tan\frac{x}{2} = \frac{1 - \sqrt{\cos\alpha}}{1 + \sqrt{\cos\alpha}} \Rightarrow \sin x = \tan^2\frac{\alpha}{2}$$

6. $T_n = \tan^{-1}\left(\frac{4}{4n^2+3}\right) = \tan^{-1}\left(\frac{1}{n^2+(3/4)}\right) = \tan^{-1}\left(\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right)$

$T_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$

$S_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow S_\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$

7. $\cos^{-1}(1-x) + m\cos^{-1} x = \frac{n\pi}{2}$

Domain $x \in [0, 1]$

$\cos^{-1}(1-x) + m\cos^{-1} x > 0 \quad (\because m > 0)$

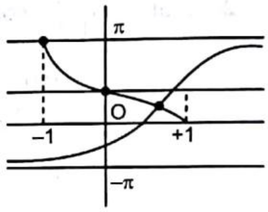
There is no solution.

8. $2\tan^{-1}(2x-1) = \cos^{-1} x$

$2x - 1 \geq 0 \qquad 1 \geq x > 0$

$x \geq \frac{1}{2}$

Only one solution



9. Put $x = 2\sin\theta,\ y = 3\cos\theta$

$\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2 = \frac{\sin\theta}{\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}} - 2 \in [-3, -1]$

$\therefore \quad \frac{\sin\theta}{\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}} - 2 = -1$ only

10. $(\cos^{-1} x)^2 - (\sin^{-1} x)^2 > 0 \Rightarrow (\cos^{-1} x + \sin^{-1} x)(\cos^{-1} x - \sin^{-1} x) > 0$

$\Rightarrow \quad \cos^{-1} x - \sin^{-1} x > 0$

$\Rightarrow \quad \frac{\pi}{2} - 2\sin^{-1} x > 0 \Rightarrow -\frac{\pi}{2} \leq \sin^{-1} x < \frac{\pi}{4} \Rightarrow -1 \leq x < \frac{1}{\sqrt{2}}$

11. $f(x) = x^2 + 7x + k(k-3) = 0$

$f(0) < 0 \quad (\because k \in (0, 3))$

$\Rightarrow$ α and β are of opposite sign.

$\tan^{-1}\alpha + \tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\beta + \tan^{-1}\left(\frac{1}{\beta}\right) = 0$

12. $f(x) = a + 2b\cos^{-1} x$

$D_f : [-1, 1]$

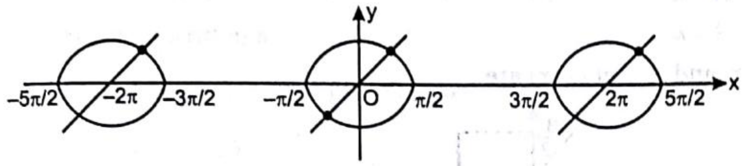
$f(x)$ is decreasing function.

$\Rightarrow \quad f(-1) = 1 \quad \Rightarrow a + 2b\pi = 1$

and $f(1) = -1 \quad \Rightarrow a = -1$

13. Let $\tan^{-1} x = t \quad \Rightarrow t^2 + \left(\frac{\pi}{2} - t\right)^2 = \frac{5\pi^2}{8}$

$\Rightarrow t = \frac{3\pi}{4}$ or $\frac{-\pi}{4} \Rightarrow \tan^{-1} x = \frac{-\pi}{4} \Rightarrow x = -1$

14. 

15. $1 \le \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x))) \le \frac{\pi}{2}$

$\sin 1 \le \cos^{-1}(\sin^{-1}(\tan^{-1} x)) \le 1$

$\cos(\sin 1) \ge \sin^{-1}(\tan^{-1} x) \ge \cos 1$

$\sin(\cos(\sin 1)) \ge \tan^{-1} x \ge \sin(\cos 1)$

$\tan(\sin(\cos(\sin 1))) \ge x \ge \tan(\sin(\cos 1))$

16. $x + \frac{1}{x} = -2\sin(\cos^{-1} y) \Rightarrow x = -1$ and $y = 0$

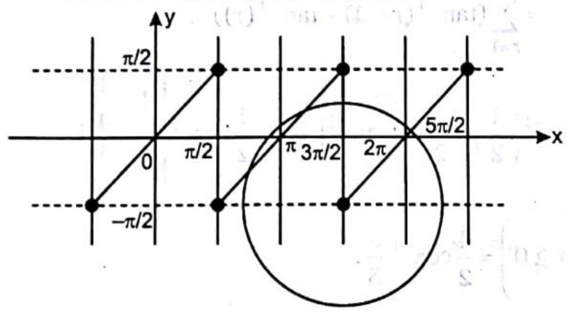
17. $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3$

$\tan^{-1} 1 + \pi + \tan^{-1}\left(\frac{5}{1-6}\right) = \pi$

18. Let $\tan^{-1} x = \theta, \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$2\theta + \cos^{-1}\cos 2\theta \Rightarrow 2\theta \le 0$

$\theta \le 0 \Rightarrow \tan^{-1} x \le 0 \Rightarrow x \le 0$

19. 

$16(x^2 + y^2) - 48\pi x + 16\pi y + 31\pi^2 = 0$

$x^2 + y^2 - 3\pi x + \pi y + \frac{31\pi^2}{16} = 0$

$$\left(x-\frac{3\pi}{2}\right)^2+\left(y+\frac{\pi}{2}\right)^2=\frac{9\pi^2}{16}$$

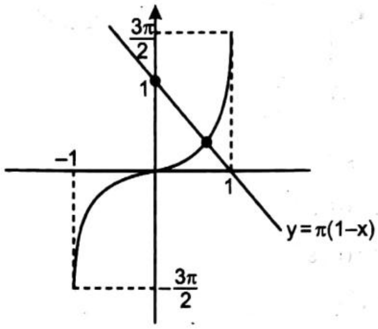
22. $\sin^{-1}(\sin 8)=3\pi-8=t$

$\tan^{-1}(\tan 8)=8-3\pi=-t$

$f(t)+f(-t)=\lambda$

$2=\lambda$

23. Graphs of $y=3\sin^{-1}x$ and $y=\pi(1-x)$ are



Clearly one point of intersection

24. $D_f:[-1,1]$

$f(x)_{\max}=\frac{\pi}{2}+6$ at $x=1$

$f(x)_{\min}=-\frac{\pi}{2}-2$ at $x=-1$

27. $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots=\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{r^2+r+1}\right)$

$$=\sum_{r=1}^{\infty}(\tan^{-1}(r+1)-\tan^{-1}(r))$$

28. $\frac{1}{2}\cos^{-1}x=\tan^{-1}\frac{(1/4)+(2/9)}{1-(1/4)\times(2/9)}=\tan^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\times 2\tan^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\cos^{-1}\left[\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right]$

$\left(\text{using } 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\text{ for } x\geq 0\right)=\frac{1}{2}\cos^{-1}\frac{3}{5}.$

29. $\tan^2(\sin^{-1}x)>1\Rightarrow -\frac{\pi}{2}<\sin^{-1}x<-\frac{\pi}{4}$ or $\frac{\pi}{4}<\sin^{-1}x<\frac{\pi}{2}$

30. $\cot^{-1}\left(\frac{1+2\times 4}{4-2}\right)+\cot^{-1}\left(\frac{1+4\times 8}{8-4}\right)+\cot^{-1}\left(\frac{1+8\times 16}{16-8}\right)+\ldots$

$= \cot^{-1}(2) - \cot^{-1}(4) + \cot^{-1}(4) - \cot^{-1}(8) + \cot^{-1}(8) - \cot^{-1}(16) + \dots$

$= \cot^{-1}(2)$

32. $\sin^{-1}(1+x)$ is defined for $x < 0$ and $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x \;\forall -1 \le x \le 1.$

The given equation is $\sin^{-1} x + \sin^{-1}(1+x) = \cos^{-1} x$

which can be written as

$$\frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \cos^{-1}(1+x) = \cos^{-1} x$$

$\Rightarrow \quad \pi - \cos^{-1}(1+x) = 2\cos^{-1} x$

$\Rightarrow \quad \cos^{-1}(-1-x) = 2\pi - \cos^{-1}(2x^2 - 1)$

$\Rightarrow \quad \cos^{-1}(-1-x) + \cos^{-1}(2x^2 - 1) = 2\pi$

$\Rightarrow \quad \cos^{-1}(-1-x) = \cos^{-1}(2x^2 - 1) = \pi$

$\Rightarrow \quad -1 - x = 2x^2 - 1 = -1$

$\Rightarrow \quad x = 0$

which implies that the total number of solutions $\sin^{-1} x + \sin^{-1}(1+x) = \cos^{-1} x$ is only one.

33. $(\sin^{-1} x)^3 - (\cos^{-1} x)^3 + (\sin^{-1} x)(\cos^{-1} x)(\sin^{-1} x - \cos^{-1} x) = \frac{\pi^3}{16}$

$(\sin^{-1} x - \cos^{-1} x)\{(\sin^{-1} x)^2 + (\cos^{-1} x)^2 + (2\cos^{-1} x \sin^{-1} x)\} = \frac{\pi^3}{16}$

$$(\sin^{-1} x - \cos^{-1} x)(\sin^{-1} x + \cos^{-1} x)^2 = \frac{\pi^3}{16}$$

$$(\sin^{-1} x - \cos^{-1} x)\frac{\pi^2}{4} = \frac{\pi^3}{16}$$

$$2\sin^{-1} x - \frac{\pi}{2} = \frac{\pi}{4}$$

$$2\sin^{-1} x = \frac{3\pi}{4}$$

$$\sin^{-1} x = \frac{3\pi}{8}$$

$$x = \sin\frac{3x}{8} \text{ or } \cos\frac{x}{8}$$

35. $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$

$$\frac{\sqrt{1+x^2} - 1}{x} = y$$

$$y' = \frac{x\frac{1}{2}\frac{2x}{\sqrt{1+x^2}} - (\sqrt{1+x^2}-1)}{x^2}$$

$$= \frac{\sqrt{1+x^2}-1}{x^2(\sqrt{1+x^2})} > 0 \quad \text{always}$$

$x \to \infty$ $\quad y \to 1$

$x \to -\infty$ $\quad y \to -1$

$\tan^{-1}(-1 \to 1)$

$\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) - \{0\}$

40. $\cos^{-1} x + \cot^{-1} x = \lambda \ \forall \ x \in [-1, 1]$

$\lambda \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

41. $x^3 + bx^2 + cx + 1 = 0$

$f(-1) = b - c < 0$

$f(0) = 1 > 0$

$\Rightarrow \quad -1 < \alpha < 0$

$\alpha = -B$

$B \in (0, 1)$

$$y = -2\tan^{-1}(\operatorname{cosec} B) - \tan^{-1}\left(\frac{2\sin B}{\cos^2 B}\right)$$

$$= -\left(\pi + \tan^{-1}\frac{2\cos B}{1-\operatorname{cosec}^2 B}\right) - \tan^{-1}\frac{2\sin B}{\cos^2 B} = -\pi$$

42. $f(x) = \frac{\pi}{2} + \cot^{-1}\{-x\}$

$\frac{\pi}{4} < \cot^{-1}\{-x\} \le \frac{\pi}{2}$

43. $\sin^{-1}(\sin 3) + \tan^{-1}(\tan 3) + \sec^{-1}(\sec 3)$

$(\pi - 3) + (3 - \pi) + 3 = 3$

44. $(2n\pi, 0)\ n \in I$

45. $f(x) = \sin^{-1}([x]-1) + 2\cos^{-1}([x]-2)$

$-1 \le [x] - 1 \le 1 \quad \Rightarrow 0 \le [x] \le 2$

$-1 \le [x] - 2 \le 1 \quad \Rightarrow 1 \le [x] \le 3 \quad \Rightarrow [x] = 1 \text{ or } 2$

Exercise-2 : One or More than One Answer is/are Correct

2. $\cos^{-1} x = \tan^{-1} x \Rightarrow x \in [0, 1]$

$$\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1} x$$

$$\Rightarrow \quad x^2 = \sqrt{1-x^2} \Rightarrow x^4 + x^2 - 1 = 0$$

$$x^2 = \frac{\sqrt{5}-1}{2}$$

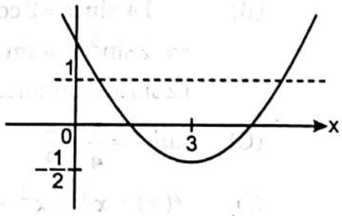
3. $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$

$a = 17,\ b = 6$

5. $\sin^{-1}\left(x^2 - 6x + \frac{17}{2}\right) = \sin^{-1} k$

where $-1 \le k \le 1$

$$y = x^2 - 6x + \frac{17}{2}$$



6. $(\sin^{-1} x - \cos^{-1} x)((\sin^{-1} x)^2 + (\cos^{-1} x)^2 + 2\sin^{-1} x \cos^{-1} x) = \frac{\pi^3}{16}$

$$\Rightarrow \sin^{-1} x - \cos^{-1} x = \frac{\pi}{4} \Rightarrow \cos^{-1} x = \frac{\pi}{8} \Rightarrow x = \cos\frac{\pi}{8}$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $a = 2\pi$

$b = -3$

2. $a = 0$

$b = 3$

Exercise-4 : Matching Type Problems

3. (A) $33n = \frac{n}{2}[2+(n-1)^2] \Rightarrow n = 9$

(B) $x \in [-1, 1] \Rightarrow \cos^{-1} x + \cot^{-1} x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

(C) $\cos\theta = |1+\sin\theta| \Rightarrow \cos\theta \geq 0$

Sq. both sides,

$\Rightarrow \quad \cos^2\theta = 1+\sin^2\theta + 2\sin\theta$

$\sin\theta = 0$ or $\sin\theta = -1$

Number of solution $= 3$

(D) $a = x(x-1)$

Possible values of a are 6, 12, 20, 30.

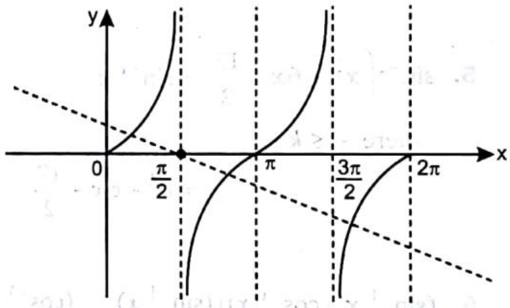
4. (A) $\tan^{-1}(3) + \tan^{-1}(-3) = 0$

(B) $1+\sin x = 2\cos^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

(C) $\tan x = \frac{\pi}{4} - \frac{\pi}{2}$



(D) $f(x) = x^3 + x^2 + 4x + 2\sin x$

$f'(x) = 3x^2 + 2x + 4 + 2\cos x > 0$

and $f(0) = 0$

Exercise-5 : Subjective Type Problems

1. $5 - 2\pi > x^2 - 4x$

$x^2 - 4x + (2\pi - 5) < 0$

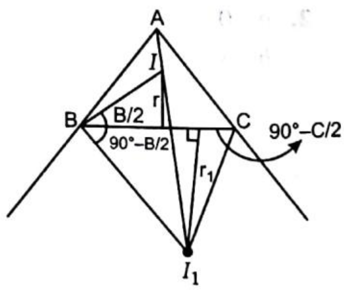
$2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi} \quad \Rightarrow \lambda = 9$

2. $\sin\frac{B}{2} = \frac{r}{IB}$

$IB = 4R\sin\frac{A}{2}\sin\frac{C}{2}$

$\sin\left(90° - \frac{B}{2}\right) = \frac{r_1}{BI_1} \quad \Rightarrow BI_1 = 4R\sin\frac{A}{2}\cos\frac{C}{2}$

$(II_1)^2 = (BI)^2 + (BI_1)^2 = 16R^2\sin^2\frac{A}{2}$ ...(1)



$$I_2 I_3 \cos\left(90° - \frac{A}{2}\right) = a \quad \text{(by using pedal triangle)}$$

$$I_2 I_3 = 4R \cos\frac{A}{2}$$

$$(I_2 I_3)^2 = 16R^2 \cos^2\frac{A}{2} \quad \ldots(2)$$

From (1) & (2) we get $\lambda = 16$

3. $2\tan^{-1}\left(\frac{1}{5}\right) - \sin^{-1}\left(\frac{3}{5}\right)$

$\tan^{-1}\left(\frac{5}{12}\right) - \sin^{-1}\left(\frac{3}{5}\right)$

$$\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}\left(\frac{3}{4}\right) = -\left(\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right)$$

$$= -\tan^{-1}\left(\frac{16}{63}\right) = -\cos^{-1}\left(\frac{63}{65}\right)$$

$\Rightarrow \quad \lambda = 65$

5. $$\sum_{n=0}^{\infty} 2\tan^{-1}\left(\frac{2}{n^2+n+4}\right) = \sum_{n=0}^{\infty} 2\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{n^2}{4}+\frac{n}{4}+1}\right)$$

$$= \sum_{n=0}^{\infty} 2\tan^{-1}\left(\frac{\left(\frac{n}{2}+\frac{1}{2}\right)-\frac{n}{2}}{\frac{n}{2}\left(\frac{n}{2}+\frac{1}{2}\right)+1}\right)$$

$$= \sum_{n=0}^{\infty} 2\left(\tan^{-1}\left(\frac{n}{2}+\frac{1}{2}\right) - \tan^{-1}\left(\frac{n}{2}\right)\right)$$

6. $\cos^{-1}(|3\log_6^2(\cos x) - 7|) = \cos^{-1}(|\log_6^2(\cos x) - 1|)$

$$|3\log_6^2(\cos x) - 7| = |\log_6^2(\cos x) - 1|$$

Let $\log_6^2(\cos x) = t$

$$|3t - 7| = |t - 1|$$

$\Rightarrow \quad t = 3$ and $t = 2$

$\Rightarrow \quad \cos x = 6^{-\sqrt{3}}$ and $6^{-\sqrt{2}}$

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Chapter 26 – Vector & 3Dimensional Geometry

26 VECTOR & 3DIMENSIONAL GEOMETRY

Exercise-1 : Single Choice Problems

1. Perpendicular distance from origin

$$d = \frac{p}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \quad d^2 = \frac{p^2}{a^2 + b^2 + c^2}$$

2. Area of triangle $= \frac{1}{2}|\vec{a} \times \vec{b}| = 3$

$$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\frac{\pi}{3} = 6 \quad \Rightarrow |\vec{a}||\vec{b}| = \frac{12}{\sqrt{3}}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\frac{\pi}{3} = 2\sqrt{3}$$

4. $|\vec{c} - \vec{a}|^2 = 8 \Rightarrow |\vec{c}|^2 - 2\,\vec{c} \cdot \vec{a} + |\vec{a}|^2 = 8 \Rightarrow |\vec{c}|^2 - 2|\vec{c}| + 1 = 0 \Rightarrow |\vec{c}|^2 = 1$

Also, $\quad \vec{a} \times \vec{b} = 2\hat{i} + 2\hat{j} + \hat{k} \quad \Rightarrow |\vec{a} \times \vec{b}| = 3$

$$\therefore \quad |(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}|\sin\frac{\pi}{6} = 3 \cdot 1 \cdot \frac{1}{2} = \frac{3}{2}$$

5.
$$\cos\theta_1 = \frac{4}{5}$$

$$\cos\theta_2 = \frac{4}{5}$$

$$\Rightarrow \quad \cos^2\theta_1 + \sin^2\theta_2 = 1$$

7. $\lambda(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = 4\sqrt{3} \qquad \left(ab\cos\frac{\pi}{3} = 1\right) \Rightarrow \quad b = 1$

$$\lambda(a^2b^2 - (\vec{a} \cdot \vec{b})^2) = 4\sqrt{3}$$

$$\lambda(4\times 1-(1)^2)=4\sqrt{3}$$

$$\lambda=\frac{4\sqrt{3}}{3}$$

8. $x(3\hat{i}+2\hat{j}+4\hat{k})+y(2\hat{i}+2\hat{k})+z(4\hat{i}+2\hat{j}+3\hat{k})=\alpha(x\hat{i}+y\hat{j}+z\hat{k})$

$$\Rightarrow \quad (3-\alpha)x+2y+4z=0$$

$$2x-\alpha y+2z=0$$

$$4x+2y+(3-\alpha)z=0$$

For non-trivial solution

$$\begin{vmatrix} 3-\alpha & 2 & 4 \\ 2 & -\alpha & 2 \\ 4 & 2 & 3-\alpha \end{vmatrix}=0$$

9. $\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}=[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$

10. $|\vec{c}|^2=4(\vec{a}\times\vec{b})^2+9b^2=4(a^2b^2-(\vec{a}\cdot\vec{b})^2)+9b^2=192$

$\vec{c}+3\vec{b}=2\vec{a}\times\vec{b}\Rightarrow c^2+9b^2+6\vec{b}\cdot\vec{c}=4(a^2b^2-(\vec{a}\cdot\vec{b})^2)$

$\Rightarrow\ 6\cdot 4\cdot\sqrt{192}\cos\theta=-288\ \Rightarrow \cos\theta=\frac{-\sqrt{3}}{2}$

11. $|\vec{a}-2\vec{b}|^2+|\vec{b}-2\vec{c}|^2+|\vec{c}-2\vec{a}|^2=5a^2+5b^2+5c^2-4(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$$=15-4(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})\le 15-4\left(\frac{-3}{2}\right)=21$$

$\because\quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\ge\frac{-3}{2}$

12. $16|\vec{a}||\vec{b}|\sin\frac{\pi}{2}=3|\vec{a}|^2+3|\vec{b}|^2+6|\vec{a}||\vec{b}|$

$\Rightarrow\quad 3a^2-10ab+3b^2=0\ \Rightarrow (3a-b)(a-3b)=0$

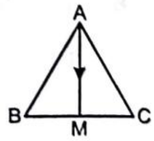
Now $\quad \overrightarrow{OC}\cdot\overrightarrow{AB}=(\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a})=|\overrightarrow{OC}||\overrightarrow{AB}|\cos\theta$

$\Rightarrow\quad \frac{b^2-a^2}{\sqrt{a^2+b^2}\sqrt{a^2+b^2}}=\cos\theta=\frac{9a^2-a^2}{9a^2+a^2}$ (using $b=3a$)

$\therefore\quad \cos\theta=\frac{4}{5}$

$\therefore \quad \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{4}{5}}{1+\frac{4}{5}}} = \frac{1}{3}$

13. $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$



14. $\begin{vmatrix} 2 & \lambda & 3 \\ 3 & 3 & 5 \\ \lambda & 2 & 2 \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 - 3\lambda + 2 = 0$

15. $(\vec{a} + \vec{b} - \vec{c}) \cdot \left[(\vec{b} + \vec{c} - \vec{a}) \times (\vec{c} + \vec{a} - \vec{b})\right]$

$(\vec{a} + \vec{b} - \vec{c}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} - \vec{c} \times \vec{b} - \vec{a} \times \vec{c} + \vec{a} \times \vec{b}) = 2(\vec{a} + \vec{b} - \vec{c}) \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a})$

$= 2([\vec{a}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}]) = 4[\vec{a}\ \vec{b}\ \vec{c}]$

16. $(\hat{a} \times \hat{b}) \times (\hat{a} + \hat{b}) = (\hat{a} \cdot (\hat{a} + \hat{b}))\,\hat{b} - (\hat{b} \cdot (\hat{a} + \hat{b}))\,\hat{a} = (1 + \hat{a} \cdot \hat{b})(\hat{b} - \hat{a})$

17. Angle between planes is angle between $\vec{n}_1$ and $\vec{n}_2$, where $\vec{n}_1 = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\vec{n}_2 = \overrightarrow{AD} \times \overrightarrow{AC}$

$\vec{n}_1 = -2\hat{i} + 4\hat{j} - 3\hat{k}, \quad \vec{n}_1 = 6\hat{i} + 3\hat{j} - 6\hat{k}$

18. $\vec{a}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{a}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{a}_3 = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ are mutually perpendicular unit vectors, then

$[\vec{a}_1\ \vec{a}_2\ \vec{a}_3] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \pm 1$

22. On solving, $Ax = C$ and $Bx = D$

$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$P = (1, 2, 3),\ Q = (3, 1, 2)$

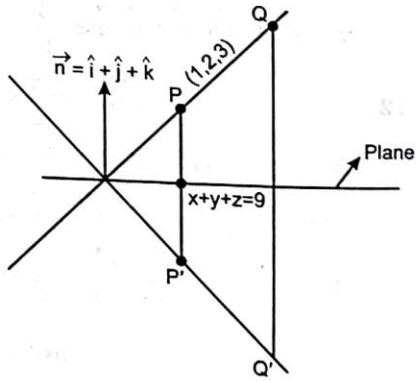
$PP' : \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1} = \lambda$

$(\lambda + 1, \lambda + 2, \lambda + 3)$ lies on plane

$3\lambda + 6 = 9 \Rightarrow \lambda = 1$

$\therefore \quad P' = (3, 4, 5)$

Similarly $\quad Q' = (5, 3, 4)$



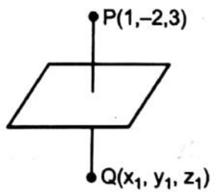
Now check the options.

23. $\overrightarrow{AM} = (\alpha - 1)\hat{i} + \hat{j}$

$\overrightarrow{BM} = (\alpha - 1)\hat{i}$

$\overrightarrow{CM} = (\alpha - 3)\hat{i} + 2\hat{j} + 2\hat{k}$ are coplanar, then $\begin{vmatrix} \alpha - 1 & 1 & 0 \\ \alpha - 2 & 0 & 0 \\ \alpha - 3 & 2 & 2 \end{vmatrix} = 0$

24. Normal vector is parallel to $\overrightarrow{PQ}$



$$\frac{x_1 - 1}{1} = \frac{y_1 + 2}{-1} = \frac{z_1 - 3}{1} = \lambda$$

$\Rightarrow \quad x_1 = \lambda + 1,\ y_1 = -2 - \lambda,\ z_1 = 3 + \lambda$

Mid point of PQ is lie on the plane

$\Rightarrow \quad \lambda = \frac{2}{3}$

$$Q\left(\frac{5}{3}, \frac{-8}{3}, \frac{11}{3}\right)$$

25. $$|\hat{a} - \hat{b}| = 1$$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

Volume of parallelopiped $= [\hat{a}\ \ \hat{b}\ \ \hat{a} \times \hat{b}] = \sin^2\theta = \frac{3}{4}$

26. Equation of line PQ

$$\frac{x - 3}{1} = \frac{y - 7}{2} = \frac{z - 1}{-6} = \lambda$$

Point $Q(3 + \lambda, 7 + 2\lambda, 1 - 6\lambda)$

If it lies on plane $3x + 2y + 11z = 9$, then

$$\lambda = \frac{25}{59}$$

27. $V_1 = [\vec{a}\ \ \vec{b}\ \ \vec{c}]$

$V_2 = [\vec{a} + \vec{b} - 2\vec{c}\ \ \ 3\vec{a} - 2\vec{b} + \vec{c}\ \ \ \vec{a} - 4\vec{b} + 2\vec{c}] = 15[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

28. Line represented by $x + ay - b = 0,\ cy + z - d = 0$ is parallel to

$$(\hat{i} + a\hat{j}) \times (c\hat{j} + \hat{k}) = a\hat{i} - \hat{j} + c\hat{k}$$

Line represented by $-x + a'y + b' = 0, c'y - z + d' = 0$ is parallel to

$$(\hat{i} - a'\hat{j}) \times (c'\hat{j} - \hat{k}) = a'\hat{i} + \hat{j} + c'\hat{k}$$

If these two lines are perpendicular, then

$$aa' + cc' = 1$$

29. Equation of line PQ

$$\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 5\hat{j} + \hat{k})$$

$\Rightarrow$ Co-ordinate of $Q(2 + \mu, 5\mu - 2, 3 + \mu)$

If point Q lies on plane, then

$$\mu = \frac{10}{27}$$

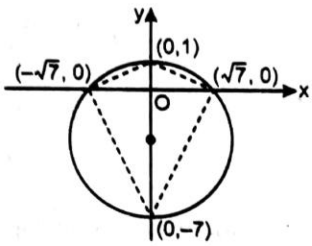
$$\overrightarrow{PQ} = \mu\hat{i} + 5\mu\hat{j} + \mu\hat{k} = \frac{10}{27}\hat{i} + \frac{50}{27}\hat{j} + \frac{10}{27}\hat{k}$$

30. $$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

$$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$\Rightarrow \quad (\vec{a} \cdot \vec{b})\vec{c} = (\vec{b} \cdot \vec{c})\vec{a}$$

31. Let $\vec{r} = x\hat{i} + y\hat{j}$

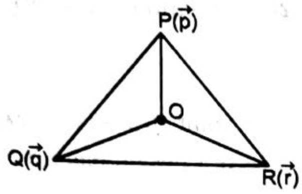


$$\vec{r} \cdot (\vec{r} + 6\hat{i}) = 7$$

$$\Rightarrow \quad x^2 + (y + 3)^2 = 16$$

Area of quadrilateral $= 8\sqrt{7}$

33. $$\frac{\frac{1}{2}|(\vec{p} - \vec{q}) \times (\vec{r} - \vec{q})|}{\frac{1}{2}|\vec{q} \times \vec{r}|} = 4$$



Also, $\vec{p} + k_1\vec{q} + k_2\vec{r} = 0$

$$\Rightarrow \quad \vec{p} = -k_1\vec{q} - k_2\vec{r} = 0$$

$$\Rightarrow \quad k_1 + k_2 + 1 = 4$$

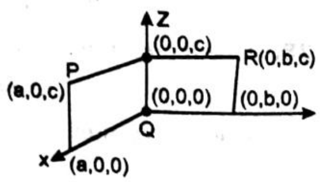
$$\Rightarrow \quad k_1 + k_2 = 3$$

34. Let length, breadth and height of rectangular box be a, b, c respectively.

$$\vec{P} = a\hat{i} + c\hat{k}$$

$$\vec{R} = b\hat{j} + c\hat{k}$$

$$\vec{O} = \frac{a}{2}\hat{i} + \frac{b}{2}\hat{j} + \frac{c}{2}\hat{k}$$



$$|\overrightarrow{OQ}||\overrightarrow{OR}|\cos\theta = \left(\frac{a}{2}\hat{i} + \frac{b}{2}\hat{j} + \frac{c}{2}\hat{k}\right) \cdot \left(\frac{a}{2}\hat{i} - \frac{b}{2}\hat{j} - \frac{c}{2}\hat{k}\right)$$

$\Rightarrow \quad \cos\theta = -\frac{1}{3}$

Similarly, $\quad \cos\phi = -\frac{1}{3}$

36. $\vec{\mathbf{r}} = a(\vec{\mathbf{m}} \times \vec{\mathbf{n}}) + b(\vec{\mathbf{n}} \times \vec{\mathbf{l}}) + c(\vec{\mathbf{l}} \times \vec{\mathbf{m}})$

where $[\vec{\mathbf{l}}\, \vec{\mathbf{m}}\, \vec{\mathbf{n}}] = 4, \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{l}} = 4a, \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{m}} = 4b, \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = 4c$

which imply that

$$\frac{a+b+c}{\vec{\mathbf{r}} \cdot (\vec{\mathbf{l}} + \vec{\mathbf{m}} + \vec{\mathbf{n}})} = \frac{1}{4}$$

37. The volume tetrahedron is given by $k = \frac{1}{6}[\vec{\mathbf{a}}\, \vec{\mathbf{b}}\, \vec{\mathbf{c}}] \Rightarrow [\vec{\mathbf{a}}\, \vec{\mathbf{b}}\, \vec{\mathbf{c}}] = 6k$

The volume of parallelepiped is given by

$$[\vec{\mathbf{a}} - \vec{\mathbf{b}} \;\; \vec{\mathbf{b}} + 2\vec{\mathbf{c}} \;\; 3\vec{\mathbf{a}} - \vec{\mathbf{c}}] = [\vec{\mathbf{a}}\, \vec{\mathbf{b}} + 2\vec{\mathbf{c}}\; 3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\, \vec{\mathbf{b}} + 2\vec{\mathbf{c}}\; 3\vec{\mathbf{a}} - \vec{\mathbf{c}}]$$

$$= [\vec{\mathbf{a}}\, \vec{\mathbf{b}}\, 3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [\vec{\mathbf{a}}\, 2\vec{\mathbf{c}}\, 3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\, \vec{\mathbf{b}}\, 3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\, 2\vec{\mathbf{c}}\, 3\vec{\mathbf{a}} - \vec{\mathbf{c}}]$$

$$= [\vec{\mathbf{a}}\, \vec{\mathbf{b}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\, 2\vec{\mathbf{c}}\, 3\vec{\mathbf{a}}] = -[\vec{\mathbf{a}}\, \vec{\mathbf{b}}\, \vec{\mathbf{c}}] - 6[\vec{\mathbf{a}}\, \vec{\mathbf{b}}\, \vec{\mathbf{c}}]$$

$$= -7[\vec{\mathbf{a}}\, \vec{\mathbf{b}}\, \vec{\mathbf{c}}]$$

Volume is 42 k.

38. We know that the equation of the plane passing through the line of intersection of planes $p_1 = 0$ and $p_2 = 0$ is

$$p_1 + \lambda p_2 = 0$$

That is,

$$(x + 2y + z - 10) + \lambda(3x + y - z - 5) = 0 \qquad \text{...(1)}$$

Since, this plane passes through the origin $(0, 0, 0)$ satisfies this equation. This implies that

$$(-10) + \lambda(-5) = 0$$

$\Rightarrow \quad \lambda = -2$

Substituting the value of λ in Eq. (1), we get

$$(x + 2y + z - 10) - 2(3x + y - z - 5) = 0$$

That is, $\quad -5x + 3z = 0$

$\Rightarrow \quad 5x - 3z = 0$

39. Let the point $P(x_p, y_p, z_p)$ be the required point.

The distance of the point from x-axis is $\sqrt{y_p^2 + z_p^2}$.

The distance from the point $(1, -1, 2)$ is

$$\sqrt{(x_p-1)^2+(y_p+1)^2+(z_p-2)^2}$$

$$\Rightarrow \quad y_p^2+z_p^2=(x_p-1)^2+(y_p+1)^2+(z_p-2)^2$$

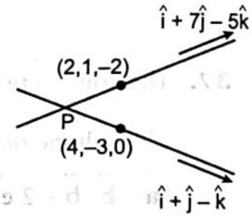
$$\Rightarrow \quad x_p^2-2x_p+2y_p-4z_p+6=0$$

Therefore, the locus of point P is

$$x^2-2x+2y-4z+6=0$$

Exercise-2 : One or More than One Answer is/are Correct

3. Point P on line L_1



$$P(2+\lambda, 1+7\lambda, -2-5\lambda)$$

Point P on line L_2

$$P(4+r, -3+r, -r) \Rightarrow \lambda=-1,\ r=-3$$

Acute angle between L_1 and L_2

$$\cos\theta=\frac{13}{15}$$

Equation of plane containing L_1 and L_2 is $x+2y+3z+2=0$

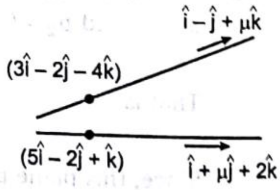
4. $\hat{a}=\hat{b}+(\hat{b}\times\hat{c})$

$\hat{a}\cdot\hat{b}=1$ and $\hat{a}\cdot\hat{c}=\hat{b}\cdot\hat{c}$

$|\hat{a}-\hat{b}|=|\hat{b}\times\hat{c}| \Rightarrow \sin\theta=0 \qquad (\because \theta=\hat{b}\hat{c})$

$|\hat{a}+\hat{b}+\hat{c}|^2=3+2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{a}\cdot\hat{c})=5+4(\hat{b}\cdot\hat{c})$

5. If these two lines are coplanar, then



$$\begin{vmatrix}1 & -1 & \mu\\ 1 & \mu & 2\\ 2 & 0 & 5\end{vmatrix}=0$$

$$\Rightarrow \quad 2\mu^2-5\mu-1=0$$

6. $\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}]+\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}]+\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]=0$

$$2\vec{a}-(\hat{i}+\hat{j}+\hat{k})=0 \quad \Rightarrow (2x-1)\hat{i}+(2y-1)\hat{j}+(2z-1)\hat{k}=0$$

$$\Rightarrow x=y=z=\frac{1}{2}$$

7. $[\vec{a}\times\vec{b}\quad \vec{c}\times\vec{d}\quad \vec{e}\times\vec{f}]=(\vec{a}\times\vec{b})\cdot[(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})]=(\vec{c}\times\vec{d})\cdot[(\vec{e}\times\vec{f})\times(\vec{a}\times\vec{b})]$

$$=(\vec{e}\times\vec{f})\cdot[(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})]$$

$$=(\vec{a}\times\vec{b})\cdot[(\vec{c}\times\vec{d})\cdot\vec{f}]\,\vec{e}-[(\vec{c}\times\vec{d}\cdot\vec{e})\times\vec{f}]$$

$$= [\vec{c}\ \vec{d}\ \vec{f}][\vec{a}\ \vec{b}\ \vec{e}] - [\vec{c}\ \vec{d}\ \vec{e}][\vec{a}\ \vec{b}\ \vec{f}]$$

Similarly, solve other 2.

8. $3(\vec{a}-\vec{b})+(\vec{b}-\vec{c})+2(\vec{c}-\vec{d})=0$

$$\frac{\overrightarrow{BC}+2\,\overrightarrow{CD}}{1+2}=\overrightarrow{BA}$$

10. $\vec{b}=2\hat{c}+\lambda\hat{a}$

$|\vec{b}|^2=4+\lambda^2+4\lambda\left(\frac{1}{4}\right)=16 \Rightarrow \lambda=-4,3$

11. $L_1: x=y=z$

$L_2: \frac{x-1}{1}=\frac{y+1}{-1}=\frac{z}{-1}$

Shortest distance $=\frac{1}{\sqrt{2}}$

Equation of plane containing line L_2 and parallel to L_1

$$y-z+1=0$$

Distance of origin from this plane $=\frac{1}{\sqrt{2}}$

12. $\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow\ [\vec{a}\,\vec{b}\,\vec{c}](\sin x+\cos y+2)=0$

$\Rightarrow\ \sin x=-1$ and $\cos y=-1$

13. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[(\vec{a}\times\vec{b})\cdot\vec{d}]\,\vec{c}-(\vec{a}\times\vec{b}\cdot\vec{c})\,\vec{d}=r\,\vec{c}+s\,\vec{d}$

where $r=[\vec{a}\ \vec{b}\ \vec{c}]$ and $s=-[\vec{a}\ \vec{b}\ \vec{c}]$ as $\vec{c}$ and $\vec{d}$ are non-collinear.

Similarly, $h=-[\vec{b}\ \vec{c}\ \vec{d}]$ and $k=[\vec{a}\ \vec{c}\ \vec{d}]$

14. Here, $\vec{\alpha}=\hat{i}+2\hat{j}$, $\vec{\beta}=2\hat{i}+a\hat{j}+10\hat{k}$ and $\vec{\gamma}=12\hat{i}+20\hat{j}+a\hat{k}$

$\therefore\ [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=\begin{vmatrix}1 & 2 & 0\\ 2 & a & 10\\ 12 & 20 & a\end{vmatrix}=a^2-24a+240>0$, for all a

$\therefore\ \vec{\alpha},\vec{\beta}$ and $\vec{\gamma}$ are non-coplanar or linearly independent for all a.

Hence, (a, b, c) is the correct answer.

19. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If $\vec{r} \times \hat{i} = \hat{j} + \hat{k} \quad \Rightarrow \quad -y\hat{k} + z\hat{j} = \hat{j} + \hat{k}$

$\Rightarrow \quad \vec{r} = x\hat{i} - \hat{j} + \hat{k}$

If $\vec{r} \times \hat{j} = \hat{i} + \hat{k} \quad \Rightarrow \quad x\hat{k} - z\hat{i} = \hat{i} + \hat{k}$

$\vec{r} = \hat{i} + y\hat{j} - \hat{k}$

20. (A) See dot product

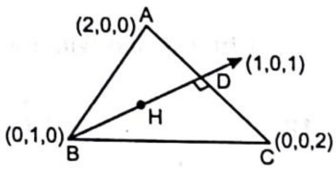
(C) $y = \ln(e^{-2} + e^x)$

$e^y - e^{-2} = e^x$

21. $(-3 - 4\lambda, 6 + 3\lambda, 2\lambda) = (-2 - 4\mu, 7 + \mu, \mu)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3



1. $AB = BC$

p.v. of $H = \hat{j} + r(\hat{i} - \hat{j} + \hat{k})$

Also, $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \quad [(r-2)\hat{i} + (1-r)\hat{j} + r\hat{k}] \cdot (-\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \quad r - 1 + 2r = 0 \quad \Rightarrow \quad r = \frac{1}{3}$

2. p.v. of $H = \frac{\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{\hat{k}}{3}$

p.v. of centroid $= \frac{2}{3}\hat{i} + \frac{\hat{j}}{3} + \frac{2\hat{k}}{3}$

p.v. of $S = \dfrac{3(\text{p.v.}) \text{ of centroid} - \text{p.v. of } \vec{H}}{2}$

y coordinate of $S = \frac{1}{6}$

3. Let $P \equiv (a, b, c)$

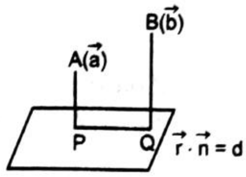
$\Rightarrow \quad (a-2)^2 + b^2 + c^2 = a^2 + (b-1)^2 + c^2 = a^2 + b^2 + (c-2)^2 = a^2 + b^2 + c^2$

$\Rightarrow \quad P = \left(1, \frac{1}{2}, 1\right)$

$PA = \frac{3}{2}$

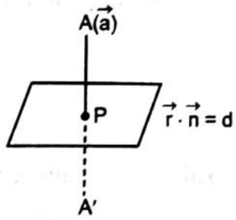
Paragraph for Question Nos. 4 to 6

4. $PQ = (\vec{b} - \vec{a})\cos\theta$ (where θ angle between AB and plane)



$$= \frac{|(\vec{b} - \vec{a}) \times \vec{n}|}{|\vec{n}|}$$

5. Equation of line AP is $\vec{r} = \vec{a} + \lambda\vec{n}$



For point $P(\vec{a} + \lambda\vec{n}) \cdot \vec{n} = d$

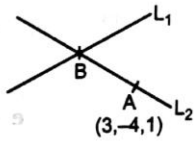
$$\lambda = \frac{d - \vec{a}\cdot\vec{n}}{|\vec{n}|^2}$$

$\therefore \quad P\left(\vec{a} + \frac{d - \vec{a}\cdot\vec{n}}{|\vec{n}|^2}\right) \quad \therefore A'$ is $\vec{a} + 2\left(\frac{d - \vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}\right)$

6. Distance $= |BQ - AP| = \left|\frac{\vec{b}\cdot\vec{n} - d}{|\vec{n}|} - \frac{\vec{a}\cdot\vec{n} - d}{|\vec{n}|}\right| = \left|\frac{(\vec{b} - \vec{n})\cdot\vec{n}}{|\vec{n}|}\right|$

Paragraph for Question Nos. 7 to 9

7. $B(3 + 2\lambda, -1 - 3\lambda, 2 - \lambda)$



d_r of $L_2 < 2\lambda, -3\lambda + 3, 1 - \lambda >$

L_2 is parallel to plane $\vec{r}\cdot(2\hat{i} + \hat{j} - \hat{k}) = 5$

$\therefore \quad 4\lambda - 3\lambda + 3 - 1 + \lambda = 0$

$2\lambda = -2 \Rightarrow \lambda = -1$

$\therefore \quad B(1, 2, 3)$

So, equation of L_2 is $\vec{r} = (3\hat{i} - 4\hat{j} + \hat{k}) + \lambda(\hat{i} - 3\hat{j} - \hat{k})$

8. Equation of plane contain L_1 & L_2 is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 0 & 3 & 1 \\ -1 & 6 & 2 \end{vmatrix} = 0$$

i.e., $(x-3)(6-6) + (y+1)(0+1) + (z-2)(0+3) = 0$

$$y + 3z - 5 = 0$$

9. Any point of L_1 is $(3 + 2\lambda, -1 - 3\lambda, 2 - \lambda)$

if on plane π, then

$2(3+2\lambda)+1(-1-3\lambda)-1(2-\lambda)=5$

$2\lambda=2 \Rightarrow \lambda=1$

$\therefore\quad Q(5,-4,1)$

if on xy plane, then $\quad 2-\lambda=0 \Rightarrow \lambda=2$

$\therefore\quad R(7,-7,0)$

$$\text{Volume of tetrahedran} = \frac{1}{6}[\overrightarrow{OA}\ \ \overrightarrow{OQ}\ \ \overrightarrow{OR}] = \frac{1}{6}\begin{vmatrix} 3 & -4 & 1 \\ 5 & -4 & 1 \\ 7 & -7 & 0 \end{vmatrix} = \frac{7}{3}$$

Paragraph for Question Nos. 10 to 11

Sol. Use crammer rule,

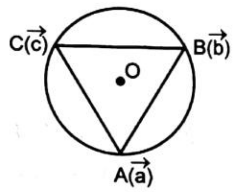
Intersect at a unique point $\Rightarrow D \neq 0$

Do not have any common point of intersection.

$\Rightarrow\quad D=0$ and atleast any one of D_x, D_y, D_z is non-zero (condition of no solution)

Paragraph for Question Nos. 12 to 14

Sol. $|\vec{\mathbf{a}}|=|\vec{\mathbf{b}}|=|\vec{\mathbf{c}}|=r$



$$\text{PV of } E: \frac{\vec{\mathbf{a}}+\left(\dfrac{\vec{\mathbf{a}}+\vec{\mathbf{b}}}{2}\right)+\vec{\mathbf{c}}}{3}$$

$$\vec{\mathbf{e}}=\frac{3\vec{\mathbf{a}}+\vec{\mathbf{b}}+2\vec{\mathbf{c}}}{6}$$

$$\text{PV of } G: \vec{\mathbf{g}}=\frac{\vec{\mathbf{a}}+\vec{\mathbf{b}}+\vec{\mathbf{c}}}{3}$$

12. $\overrightarrow{\mathbf{OE}}\cdot\overrightarrow{\mathbf{CD}}=0 \Rightarrow \left(\dfrac{3\vec{\mathbf{a}}+\vec{\mathbf{b}}+2\vec{\mathbf{c}}}{6}\right)\cdot\left(\dfrac{\vec{\mathbf{a}}+\vec{\mathbf{b}}}{2}-\vec{\mathbf{c}}\right)=0$

$\Rightarrow\quad \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}-\vec{\mathbf{c}})=0$

$\therefore\quad \overrightarrow{\mathbf{OA}}\perp\overrightarrow{\mathbf{BC}}$

$\therefore\quad \Delta ABC$ must be isosceles with base BC.

$\therefore\quad |\overrightarrow{\mathbf{AC}}|=|\overrightarrow{\mathbf{AB}}|$

13. $\overrightarrow{GE}\cdot\overrightarrow{CD}=0 \Rightarrow \left(\dfrac{3\vec{a}+\vec{b}+2\vec{c}}{6}-\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}\right)\cdot\left(\dfrac{\vec{a}+\vec{b}}{2}-\vec{c}\right)=0$

$\Rightarrow\ (\vec{a}-\vec{b})\cdot\vec{c}=0 \Rightarrow \overrightarrow{AB}\perp\overrightarrow{OC}$

$\therefore$ *ABC* must be isosceles with base *AB*.

$\therefore$ Circumcentre and centroid lie on median through *C*.

$\therefore$ Orthocenter also lie on median through *C*.

14. $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AB}\times\overrightarrow{AC}]=(\overrightarrow{AB}\ \overrightarrow{AC})^2$

$(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})^2$

$[\overrightarrow{AE}\ \overrightarrow{AG}\ \overrightarrow{AE}\times\overrightarrow{AG}]=(\overrightarrow{AE}\times\overrightarrow{AG})^2=\left\{\dfrac{-1}{18}(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\right\}^2$

$=\dfrac{1}{324}(\overrightarrow{AB}\times\overrightarrow{AC})^2$

Paragraph for Question Nos. 15 to 16

15. $D(3,-1,2)$ *AB* lies along $(0,1,2)$

CD lies along $(3,-2,0)$

Equation of plane containing *AB* line

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & 1 & 2 \\ 2 & -2 & 0 \end{vmatrix}=2(x-1)+2(y-1)-(z-1)=0$$

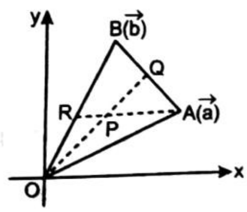
Containing *CD* line $2(x-1)+2(y-1)-(z-2)=0$

16. $r=(3,-1,2)+d(1,0,0)$

Equation of *ABC* plane is $x=1$.

Paragraph for Question Nos. 17 to 18

17. $R\left(\dfrac{2\vec{b}}{5}\right)$ and $Q\left(\dfrac{3\vec{b}+2\vec{a}}{5}\right)$



$$\frac{\mu\left(\dfrac{3\vec{b}+2\vec{a}}{5}\right)}{\mu+1}=\frac{\lambda\vec{a}+1\left(\dfrac{2\vec{b}}{5}\right)}{\lambda+1}$$

$\Rightarrow\ \dfrac{2\mu}{5(\mu+1)}=\dfrac{\lambda}{\lambda+1}$ and $\dfrac{3\mu}{5(\mu+1)}=\dfrac{2}{5(\lambda+1)} \Rightarrow \mu=\dfrac{10}{9}$

18. $\text{Ar}(\Delta OPA) = \frac{1}{2}\left|\overrightarrow{\mathbf{OP}} \times \overrightarrow{\mathbf{OA}}\right| = \frac{1}{2}\left[\frac{2}{19}(3\vec{\mathbf{b}} + 2\vec{\mathbf{a}}) \times \vec{\mathbf{a}}\right] = \frac{3}{19}(\vec{\mathbf{b}} \times \vec{\mathbf{a}})$

$$\text{Ar}(PQBR) = \frac{1}{2}\left|\overrightarrow{\mathbf{OQ}} \times \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OP}} \times \overrightarrow{\mathbf{OR}}\right| = \frac{1}{2}\left[\left(\frac{3\vec{\mathbf{b}} + 2\vec{\mathbf{a}}}{5}\right) \times \vec{\mathbf{b}} - \frac{2}{19}(3\vec{\mathbf{b}} + 2\vec{\mathbf{a}}) \times \frac{2\vec{\mathbf{b}}}{5}\right]$$

$$= \frac{3}{19}(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$

Exercise-4 : Matching Type Problems

1. (A) Line $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ is along the vector $\vec{a} = -2\hat{i} + 3\hat{j} - \hat{k}$ and line $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ is along the vector $\vec{b} = \hat{i} + \hat{j} + \hat{k}$. Here $\vec{a} \perp \vec{b}$.

Also, $\begin{vmatrix} 3-1 & -1-(-2) & 1-0 \\ -2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

(B) The direction ratios of the line $x - y + 3z - 4 = 0 = 2x + y - 3z + 5 = 0$ are

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = \hat{i} + 7\hat{j} + 3\hat{k}$$

Hence, the give two lines are parallel.

(C) The given lines are

$(x = t - 3, y = 2t + 1, z = -3t - 2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$,

or $\frac{x+3}{1} = \frac{y-1}{-2} = \frac{z+2}{-3}$ and $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+9}{-1}$

The lines are perpendicular as $(1)(1) + (-2)(2) + (-3)(-1) = 0$

Also, $\begin{vmatrix} -3-1 & 1-3 & -2-(-9) \\ 1 & -2 & -3 \\ 1 & 2 & -1 \end{vmatrix} = 0$

Hence, the lines are intersecting.

(D) The given lines are $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = (-\hat{i} - 2\hat{j} + 5\hat{k}) + s(\hat{i} - 2\hat{j} + \frac{3}{4}\hat{k})$.

$$\begin{vmatrix} 1-(-1) & 3-(-2) & -1-5 \\ 2 & -1 & -1 \\ 1 & -2 & 3/4 \end{vmatrix} = 0$$

Hence, the lines are coplanar and hence intersecting (as the lines are not parallel).

2. (A) If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular, then $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$

$$= (|\vec{a}||\vec{b}||\vec{c}|)^2 = 16$$

(B) Given $\vec{a}$ and $\vec{b}$ are two unit vectors, *i.e.*, $|\vec{a}| = |\vec{b}| = 1$ and angle between them is $\frac{\pi}{3}$.

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \Rightarrow \sin\frac{\pi}{3} = |\vec{a}\times\vec{b}|;\quad \frac{\sqrt{3}}{2} = |\vec{a}\times\vec{b}|$$

Now $[\vec{a}\ \ \vec{b}+\vec{a}\times\vec{b}\ \ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{b}] + [\vec{a}\ \vec{a}\times\vec{b}\ \vec{b}] = 0 + [\vec{a}\ \vec{a}\times\vec{b}\ \vec{b}]$

$$= (\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a}) = -|\vec{a}\times\vec{b}|^2 = -\frac{3}{4}$$

(C) If $\vec{b}$ and $\vec{c}$ are orthogonal, $\vec{b}\cdot\vec{c} = 0$

Also, it is given that $\vec{b}\times\vec{c} = \vec{a}$

Now $[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}] = [\vec{a}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}] + [\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot\vec{a} = |\vec{a}|^2 = 1$$

(because $\vec{a}$ is a unit vector)

(D) $[\vec{x}\ \ \vec{y}\ \ \vec{a}] = 0$

Therefore, $\vec{x}, \vec{y}$ and $\vec{a}$ are coplanar.

$$[\vec{x}, \vec{y}\ \vec{b}] = 0$$

Therefore, $\vec{x}, \vec{y}$ and $\vec{b}$ are coplanar.

Also, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Therefore, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

From (i), (ii) and (iii)

$\vec{x}, \vec{y}$ and $\vec{c}$ are coplanar. Therefore, $[\vec{x}, \vec{y}\ \vec{c}] = 0$

Exercise-5 : Subjective Type Problems

1. Line L is the shortest distance line of given lines.

2. $[\hat{a}\ \hat{b}\ \hat{c}]=[\hat{b}\times\hat{c}\ \ \hat{c}\times\hat{a}\ \ \hat{a}\times\hat{b}]=[\hat{a}\ \hat{b}\ \hat{c}]^2$

$\Rightarrow \qquad [\hat{a}\ \hat{b}\ \hat{c}]=1$

Projection of $\hat{b}+\hat{c}$ on $\hat{a}\times\hat{b}=\dfrac{(\hat{b}+\hat{c})\cdot(\hat{a}\times\hat{b})}{|\hat{a}\times\hat{b}|}=\dfrac{[\hat{a}\ \hat{b}\ \hat{c}]}{|\hat{a}\times\hat{b}|}$

3. Let $l=m=n=\dfrac{1}{\sqrt{2}}$

4. $\overrightarrow{OC}=\alpha^2(\vec{a}+\vec{b})^2+\beta^2(\vec{a}\times\vec{b})^2+2\alpha\beta[\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{a}\times\vec{b})]$

$\Rightarrow \qquad 1=\alpha^2\left(1+1+2\cdot1\cdot1\dfrac{1}{2}\right)+\beta^2\cdot1\cdot1\left(\dfrac{\sqrt{3}}{2}\right)^2+0 \qquad \ldots(1)$

Also, $\qquad \overrightarrow{OB}\cdot\overrightarrow{OC}=|\overrightarrow{OB}|\cdot|\overrightarrow{OC}|\cos\dfrac{\pi}{3}$

$\Rightarrow \qquad \alpha\cdot1\cdot1\cdot\dfrac{1}{2}+\alpha\cdot1=\dfrac{1}{2}\ \Rightarrow \alpha=\dfrac{1}{3} \qquad \ldots(2)$

From (1) and (2), $\qquad \beta^2=\dfrac{8}{9}$

5. $\vec{v}_{n+1}-\vec{v}_n=\left(\begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}\right)^{n+1}\vec{v}_0$

$\vec{v}_2-\vec{v}_1=\begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}\vec{v}_0$

$\vec{v}_3-\vec{v}_2=\begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}^3\vec{v}_0$

$\vec{v}_n-\vec{v}_{n-1}=\begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}^n\vec{v}_0$

Adding all the equations,

$\vec{v}_n-\vec{v}_0=(A+A^2+A^3+\ldots\ldots A^n)\,\vec{v}_0$

where $A=\begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}\Rightarrow \vec{v}_n=(I+A+A^2+\ldots\ldots)\,\vec{v}_0$

6. Let $$B = A - \frac{1}{3}A^2 + \frac{1}{9}A^3 - \frac{1}{27}A^4 + \ldots\ldots$$

$$-\frac{AB}{3} = -\frac{A^2}{3} + \frac{1}{9}A^3 - \frac{1}{27}A^4 + \ldots\ldots$$

$$\left(I + \frac{A}{3}\right)B = A$$

$$B = \frac{1}{3}(3I + A)^{-1}A$$

7. $\det M_n = \sum_{k=0}^{n}\left(\frac{1}{(2k+1)!} - \frac{1}{(2k+2)!}\right) = \frac{1}{1!} - \frac{1}{(2n+2)!}$

8. $$|\vec{a} + \vec{b}| = \sqrt{3}$$

$\Rightarrow$ Squaring both sides

$\Rightarrow$ $$\vec{a} \cdot \vec{b} = \frac{1}{2}$$

$$\vec{c} = \vec{a} + 2\vec{b} - 3\vec{a} \times \vec{b}$$

$\Rightarrow$ $$\vec{a} \cdot \vec{c} = 2 \ \& \ \vec{b} \cdot \vec{c} = \frac{5}{2}$$

$$p = |(\vec{a} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}|$$

$$p = \sqrt{\left|2\vec{b} - \frac{5}{2}\vec{a}\right|^2}$$

$$p = \frac{\sqrt{21}}{2} \Rightarrow [p] = 2$$

9. $$\vec{r} = (\vec{a} \times \vec{b})\sin x + (\vec{b} \times \vec{c})\cos y + 2(\vec{c} \times \vec{a})$$

$$\vec{r} \cdot \vec{a} = [\vec{b}\ \vec{c}\ \vec{a}]\cos y$$

$$\vec{r} \cdot \vec{b} = 2[\vec{c}\ \vec{a}\ \vec{b}]$$

$$\vec{r} \cdot \vec{c} = \sin x[\vec{a}\ \vec{b}\ \vec{c}]$$

$\Rightarrow$ $\sin x + \cos y + 2 = 0$

$\Rightarrow$ $\sin x = -1$ and $\cos y = -1$

$x = -\frac{\pi}{2}$ $\qquad y = \pi$

10. New equation of plane : $4x + 7y + 4z + 81 + \lambda(5x + 3y + 10z - 25) = 0$

$$(4 + 5\lambda)4 + (7 + 3\lambda)7 + (4 + 10\lambda)4 = 0$$

$\Rightarrow$ $$\lambda = -1$$

$\Rightarrow$ Equation of plane : $x - 4y + 6z - 106 = 0$

$$\text{distance} = \frac{106}{\sqrt{53}} = \sqrt{212}$$

18. $\vec{\omega} \times \vec{\mu} = \vec{\nu}$

$$\vec{\nu} \cdot (\vec{\omega} \times \vec{\mu}) = \vec{\nu} \cdot \vec{\nu} = 1$$

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